HOMOTOPY PROPERTY OF DEFINABLE FIBER BUNDLES

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Received September 4, 2002

ABSTRACT. Let $\eta = (E, p, X, F, K)$ be a definable fiber bundle over a definable set X with fiber F and structure group K and $f, h : Y \to X$ definable maps between definable sets. We prove that if f and h are homotopic and Y is compact, then the induced definable fiber bundles $f^*(\eta)$ and $h^*(\eta)$ are definably fiber bundle isomorphic.

Let G be a compact definable group and X a compact definable G set. We prove that if X has only one orbit type, then $(X, p, X/G, G/H, N(H)/H)$ is a definable fiber bundle, where $p : X \to X/G$ denotes the orbit map.

1. INTRODUCTION

Let $\mathcal{M}$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$". General references on o-minimal structures are [5], [7], see also [16]. Further properties and constructions of them are studied in [6], [8], [15]. A definable category is a generalization of the semialgebraic category, and the definable category on $\mathcal{R}$ coincides with the semialgebraic one.

The homotopy property of semialgebraic vector bundles is established in 12.7.7 [1] and it is studied in 2.10 [14] of equivariant fiber bundles. An equivariant version of 12.7.7 [1] is studied in [3], definable G sets and definable G maps are studied in [9], and definable C^rG manifolds and definable C^rG vector bundles are studied in [12], [11], [10].

In this paper, we use a definable space as in the sense of [5]. Every definable set is a definable space in this sense. Definable maps between definable spaces are assumed to be continuous.

Theorem 1.1. *Let $\eta = (E, p, X, F, K)$ be a definable fiber bundle over a definable set X with fiber F and structure group K. If two definable maps $f, h : Y \to X$ between definable sets are homotopic and Y is compact, then $f^*(\eta)$ and $h^*(\eta)$ are definably fiber bundle isomorphic.*

Let X and Y be definable sets. Two definable maps $f, h : X \to Y$ are called *definably homotopic* if there exists a definable map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$. By 1.2 [9], if two definable maps between definable sets are homotopic, then they are definably homotopic. Hence two definable maps in Theorem 1.1 are definably homotopic.

Theorem 1.2. *Let G be a compact definable group and X a compact definable G set. If X has only one orbit type G/H, then $(X, p, X/G, G/H, N(H)/H)$ is a definable fiber*

2000 *Mathematics Subject Classification.* 14P10, 57R22, 03C64.
Keywords and Phrases. Definable sets, definable fiber bundles, definable vector bundles, definably homotopic, o-minimal.

bundle, where $p : X \to X/G$ *is the orbit map and* $N(H)$ *denotes the normalizer of* H *in* G.

A definable C^∞ version of Theorem 1.2 is known in 1.4 [11].

Proposition 1.3. *Let* G *be a definable group and* H *a definable subgroup of* G. *Then* $(G, p, G/H, H)$ *is a principal definable fiber bundle, where* $p : G \to G/H$ *denotes the orbit map.*

Note that in Proposition 1.3, if G is a compact definable subgroup of some $GL_n(\mathbb{R})$, then Proposition 1.3 is a corollary of Theorem 1.2.

2. Definable fiber bundles and proof of results

A *definable set* means a definable subset of some Euclidean space $\mathbb{R}^n$. A group G is a *definable group* if G is a definable set such that the group operations $G \times G \to G$ and $G \to G$ are definable. A subgroup H of a definable group G is a *definable subgroup* if it is a definable subset of G. By 2.12 [11], every definable subgroup of a definable group is closed and a closed subgroup of a definable group is not necessarily definable. A group homomorphism (resp. An group isomorphism) between definable groups is a *definable group homomorphism* (resp. a *definable group isomorphism*) if it is a definable map. A *representation map* of a definable group G is a definable group homomorphism from G to some $O_n(\mathbb{R})$. A *representation* of G means some $\mathbb{R}^n$ with the linear action induced by a representation map $G \to O_n(\mathbb{R})$. In this paper, we assume that every representation of G is orthogonal. A G invariant definable subset of a representation of a definable group G is called a *definable* G *set*.

Let G be a definable group. A *definable set with a definable* G *action* is a pair (X, θ) consisting of a definable set X and a group action $\theta : G \times X \to X$ such that θ is a definable map. We simply write X instead of (X, θ). Clearly a definable G set is a definable set with a definable G action.

A *definable space* is an object obtained by pasting finitely many definable sets together along open definable subsets, and definable maps between definable spaces are defined similarly (see Chap. 10 [5]). Definable spaces are generalizations of semialgebraic spaces in the sense of [4].

Definition 2.1. (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

(a) The total space E is a definable space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p : E \to X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of η such that each U_i is a definable open subset of X, $\{U_i\}_i$ is a finite open covering of X. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where π_i stands for the projection $U_i \times F \to F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations *definable*.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\overline{f} : E \to E'$ is said to be a *definable fiber bundle morphism* if the following two conditions are satisfied:
   (a) There exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.
   (b) For any i, j such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in K, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

   A definable fiber bundle morphism $\overline{f} : E \to E'$ is called a *definable fiber bundle isomorphism* if $X = X'$, $f = id_X$ and there exists a definable fiber bundle morphism $\overline{f'} : E' \to E$ such that $f' = id_X$, $\overline{f} \circ \overline{f'} = id$, and $\overline{f'} \circ \overline{f} = id$. We say that η is *definably trivial* if η is definably fiber bundle isomorphic to the trivial bundle $(X \times F, proj, X, F, K)$, where $proj : X \times F \to X$ denotes the projection onto the first factor.

(3) A continuous section $s : X \to E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable section* if for any i, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

(4) We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal definable fiber bundle* if $F = K$ and the K action on F is defined by the multiplication of K.

To prove Theorem 1.1, we need the following three results.

Lemma 2.2. *Let A be a definable set, $X_1 = \{(x_1, x_2) \in A \times [0,1] | f_1(x_1) < x_2 \leq f_2(x_1)\}, X_2 = \{(x_1, x_2) \in A \times [0,1] | f_2(x_1) \leq x_2 < f_3(x_1)\}$ and $\eta = (E, p, X, F, K)$ a definable fiber bundle over $X = X_1 \cup X_2$, where $f_i : A \to [0,1]$, $(1 \leq i \leq 3)$, are definable functions with $f_1 < f_2 < f_3$. If $\eta|X_1$ and $\eta|X_2$ are definably trivial, then η is definably trivial.*

Proof. Let $u_i : X_i \times F \to p^{-1}(X_i)$, $(i = 1, 2)$, be definable fiber bundle isomorphisms, and $w_i := u_i|(X_1 \cap X_2) \times F$, $(i = 1, 2)$. Then $h := w_2^{-1} \circ w_1 : (X_1 \cap X_2) \times F \to (X_1 \cap X_2) \times F$ is a definable fiber bundle isomorphism. Hence there exists a definable map $l : X_1 \cap X_2 \to K$ such that $h(x, y) = (x, l(x)y)$, where $(x, y) \in (X_1 \cap X_2) \times F$. Let $i_A : A \to X_1 \cap X_2, i_A(a) = (a, f_2(a))$ and $l' : A \to K, l' = l \circ i_A$. We extend h to a definable fiber bundle isomorphism

$$\tilde{h} : X_2 \times F \to X_2 \times F, \tilde{h}(x_1, x_2, y) = (x_1, x_2, l'(x_1)y).$$

Then two definable fiber isomorphisms $u_1 : X_1 \times F \to p^{-1}(X_1)$ and $u_2 \circ \tilde{h} : X_2 \times F \to p^{-1}(X_2)$ coincide on $(X_1 \cap X_2) \times F$, and $X_1 \times F$ and $X_2 \times F$ are closed in $(X_1 \cup X_2) \times F = X \times F$. Therefore the gluing map provides the desired isomorphism. □

Lemma 2.3. *Let X be a compact definable set and $\eta = (E, p, X \times [0,1], F, K)$ a definable fiber bundle over $X \times [0,1]$. Then there exists a finite definable open covering $\{U_i\}_i$ of X such that each $\eta|(U_i \times [0,1])$ is definable trivial.*

Proof. For any $(x, t) \in X \times [0,1]$, there exists a definable open neighborhood $U_{x,t}$ of (x, t) in $X \times [0,1]$ such that $\eta|U_{x,t}$ is definably trivial. Since $[0,1]$ is compact and by

Lemma 2.2, one can find a definable open neighborhood U_x of x in X such that $\eta|U_x \times [0,1]$ is definably trivial. Since X is compact and $\{U_x\}_{x\in X}$ is a definable open covering of X, there exists the required definable open covering of X. □

Theorem 2.4. *Let X be a compact definable set, $r : X\times[0,1] \to X\times[0,1], r(x,t) = (x,1)$ and $\eta = (E, p, X \times [0,1], F, K)$ a definable fiber bundle over $X \times [0,1]$. Then there exists a definable fiber bundle morphism $\phi : E \to E$ with $p \circ \phi = r \circ p$.*

Proof. By Lemma 2.3 and since X is compact, we can find a finite definable open covering $\{U_i \times [0,1]\}_{i=1}^n$ of $X \times [0,1]$ such that each $\eta|(U_i \times [0,1])$ is definably trivial. Since $\{U_i\}_{i=1}^n$ is a definable open covering of X and by 6.3.7 [5], there exists a definable partition of unity, namely there exist definable functions $f_1, \ldots, f_n : X \to [0,1]$ such that:

1. The support of each f_i is contained in U_i.
2. $\max_{1\le i\le n} f_i(x) = 1$ for all $x \in X$.

Let $\{h_i : U_i \times [0,1] \times F \to p^{-1}(U_i \times [0,1])\}_{i=1}^n$ be definable local trivializations of η. Define

$$(u_i, r_i) : (E, X \times [0,1]) \to (E, X \times [0,1]), 1 \le i \le n,$$

$$r_i(x,t) = \begin{cases} (x, \max(f_i(x), t)), & (x,t) \in U_i \times [0,1] \\ (x,t), & \text{otherwise} \end{cases},$$

$$u_i(h_i(x,t,y)) = h_i(x, \max(f_i(x), t), y), \text{ for any } (x,t,y) \in U_i \times [0,1] \times F,$$

$$u_i \text{ is the identity outside } p^{-1}(U_i \times [0,1]).$$

Then $r = r_n \circ \cdots \circ r_1$. Therefore $\phi := u_n \circ \cdots \circ u_1 : E \to E$ is the required definable fiber bundle morphism. □

By Lemma 2.2, 2.3 and Theorem 2.4, we have Theorem 1.1.

Corollary 2.5. *Every definable fiber bundle over a compact contractible definable set is definably trivial.*

Let G be a compact definable group and X a definable set with a definable G action. Then by 10.2.18 [5], the orbit space X/G exists as a definable proper quotient, namely X/G is a definable set and the orbit map $p : X \to X/G$ is a surjective proper definable map.

Definition 2.6. Let G be a compact definable group, X a definable set with a definable G action and $x \in X$. A G_x invariant definable subset S of X is a *definable slice* at x in X if GS is a definable open neighborhood of the orbit $G(x)$ of x in X, $G \times_{G_x} S$ is a definable set with the standard definable G action $G \times (G \times_{G_x} S) \to G \times_{G_x} S, (g, [g', s]) \mapsto [gg', s]$, and the map $G \times_{G_x} S \to GS \subset X$ defined by $[g, s] \mapsto gs$ is a definable G homeomorphism.

Let G be a compact Lie group. Then there exists a slice at a point of a completely regular G space (e.g. II.5.3 [2]). The following is a definable version of it.

Theorem 2.7. *Let G be a compact definable group, X a definable G set and $x \in X$. Then there exists a definable slice S at x in X.*

Proof. By the definition of definable G sets, X is contained in a representation Ω of G as a G invariant definable subset. By 2.12 [11], G is definably group isomorphic to a definable C^1 group G', and a definable group homomorphism between definable C^1 groups is a definable C^1 group homomorphism. Thus Ω is a representation of G'. Hence at the beginning, we may assume that G is a compact definable C^1 group.

Since G is a compact definable C^1 group and by 2.22 [11], the orbit $G(x)$ of x is a definable C^1G submanifold of Ω. By 1.2 [10], one can find a definable C^1G tubular neighborhood of $G(x)$ in Ω. Then restricting it to X, we have a G invariant definable open neighborhood U of $G(x)$ in X and a definable G retraction q from U to $G(x)$, namely $q : U \to G(x)$ is a definable G map with $q|G(x) = id_{G(x)}$. Let $S := q^{-1}(x)$. Then S is a definable G_x set and $U = GS$. By II.4.2 [2], $f : G \times_{G_x} S \to GS$ ($\subset X$) defined by $f([g, s]) = gs$ is a G homeomorphism. On the other hand, the map $k : G \times S \to GS$ defined by $k(g, s) = gs$ and the projection $\pi : G \times S \to G \times_{G_x} S$ are definable maps. Since the graph of f is the image of that of k by $\pi \times id_{GS}$, f is a definable G homeomorphism. □

Note that the key of the proof of Theorem 2.7 is existence of a definable G retraction q from a G invariant definable open neighborhood U of $G(x)$ in X to $G(x)$.

Proof of Theorem 1.2. For any $x \in X$, by Theorem 2.7 and since X has only one orbit type G/H, there exists a definable slice $f_x : G \times_H S_x \to GS_x \subset X$ at x such that S_x has the trivial H action. Hence $G \times_H S_x$ is definably G homeomorphic to $S_x \times G/H$. Thus we may assume that f_x is a definable G homeomorphism from $S_x \times G/H$ onto GS_x. Let $W_x = p(S_x)$. Then $h_x := p|S_x : S_x \to W_x$ is a definable homeomorphism.

Since X is compact and $\{GS_x\}_{x\in X}$ is a definable open covering of X, there exists a finite subcover $\{GS_{x_i}\}_{i=1}^n$ of $\{GS_x\}_{x\in X}$. Note that we only use compactness of X to obtain a finite cover of sets of the form GS_x.

Let $q_i : S_{x_i} \times G/H \to G/H$ denote the projection onto the second factor. We define

$$\phi_i : p^{-1}(W_{x_i}) \to W_{x_i} \times G/H, \phi_i(x) = (p(x), q_i \circ f_{x_i}^{-1}(x)).$$

Then ϕ_i is a definable homeomorphism whose inverse is $\phi_i^{-1}(w_i, gH) = f_{x_i}(h_{x_i}^{-1}(w_i), gH)$. Let

$$\theta_{ij} : W_{x_i} \cap W_{x_j} \to G/H, \theta_{ij} = q_j \circ f_{x_j}^{-1} \circ f_{x_i} \circ k_{x_i} \circ h_{x_i}^{-1}|W_{x_i} \cap W_{x_j},$$

where k_{x_i} denotes the inclusion $S_{x_i} \to S_{x_i} \times G/H$. Then θ_{ij} is a definable map. For any $s_i \in S_{x_i}$ with $h_{x_i}(s_i) = h_{x_j}(s_j)$, $\theta_{ij}(h_{x_i}(s_i)) = q_j \circ f_{x_j}^{-1} \circ f_{x_i} \circ k_{x_i}(s_i) = q_j \circ f_{x_j}^{-1} \circ f_{x_i}(s_i, eH)$. Thus $f_{x_j}^{-1} \circ f_{x_i}(s_i, eH) = (s_j, \theta_{ij}(h_{x_i}(s_i))$. Let $gH = \theta_{ij}(h_{x_i}(s_i))$. Then since f_{x_i} is a definable G imbedding, we have $H = G_{(s_i,eH)} = G_{(s_j,gH)} = G_{[g,s_j]} = gH_{s_j}g^{-1} = gHg^{-1}$, where we identified $S_{x_j} \times G/H$ with $G \times_H S_{x_j}$ in the third equality. Hence $g \in N(H)$. Thus θ_{ij} is a definable map from $W_{x_i} \cap W_{x_j}$ to $N(H)/H$. On the other hand, using the fact that f_{x_i} and q_j are definable G maps, $\phi_j \circ \phi_i^{-1} : (W_{x_i} \cap W_{x_j}) \times G/H \to (W_{x_i} \cap W_{x_j}) \times G/H$ satisfies $\phi_j \circ \phi_i^{-1}(w, gH) = (w, gH\theta_{ij}(w))$. Thus θ_{ij} is the transition function of ϕ_i and ϕ_j. Therefore $(X, p, X/G, G/H, N(H)/H)$ is a definable fiber bundle. □

Let H be a definable subgroup of a definable group G. Then by [15], the orbit space G/H becomes a definable object, namely G/H is a definable set and the orbit map $p : G \to G/H$ is a definable map. Moreover G/H becomes a definable set with the

standard definable G action $G \times G/H \to G/H, (g, g'H) \mapsto gg'H$. By a way similar to the proof of 2.25 [13], we have the following.

Proposition 2.8. *Let G be a definable group and H a definable subgroup of G. If there exists a definable open subset U of G/H with a definable section $U \to G$ of the orbit map $p : G \to G/H$ such that its finitely many translates cover G/H, then $(G, p, G/H, H)$ is a principal definable fiber bundle.*

Proof of Proposition 1.3. By the proof of 1.3 [11], there exists a definable open subset U of G/H with a definable section $U \to G$ of p such that its finitely many translates cover G/H. Thus by Proposition 2.8, $(G, p, G/H, H)$ is a principal definable fiber bundle. □

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