DEFINABLE G CW COMPLEX STRUCTURES OF DEFINABLE G SETS AND THEIR APPLICATIONS

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ABSTRACT. Let G be a compact definable group. We prove that every pair of a definable G set and its closed definable G subset admits simultaneously definable G CW complex structures. As its applications, we prove that a canonical map from the set of definable G homotopy classes of definable G maps between definable G sets to that of G homotopy classes of continuous G maps between them is bijective. Moreover we prove that if G is a finite group, then the set of G vector bundle isomorphism classes of G vector bundles over a definable G set corresponds bijectively to that of definable G vector bundle isomorphism classes of definable G vector bundles.

1. INTRODUCTION

Let G be a compact Lie group and X a semialgebraic G set. Then X admits a semialgebraic G CW complex structure [15], and semialgebraic G sets and semialgebraic G maps are studied in [14]. Fundamental properties of semialgebraic sets and semialgebraic maps between them are collected in [3].

An o-minimal category expanding the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers is larger than the semialgebraic category. Definable sets and definable maps between definable sets in an o-minimal structure are generalizations of semialgebraic sets and semialgebraic maps between semialgebraic sets. Many remarkable results on o-minimal categories are known (e.g. [5], [6], [7], [8], [9], [16], [17]).

In this paper, we are concerned with definable G CW complex structures of definable G sets and their applications in an o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ of $\mathcal{R}$. The term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{M}$". Detailed properties of definable sets and maps are collected in [5], and some of good references of o-minimal structures are [5], [8].

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m$) is a definable set. Note that if $\mathcal{M} = \mathcal{R}$, then a definable set is a semialgebraic set and a definable map between definable sets is a semialgebraic map [19]. A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.

Let G be a compact definable group. A *definable G set* means a G invariant definable subset of some representation of G. A *definable G CW complex* is a finite G CW complex such that the characteristic map of each G cell is a definable G map (see Definition 2.2). Note that a G CW subcomplex of a definable G CW complex is a definable G CW complex itself.

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Theorem 1.1. *Let G be a compact definable group. Let X be a definable G set and Y a closed definable G subset of X. Then there exist a definable G CW complex Z in a representation Ω of G, a G CW subcomplex W of Z, and a definable G map $f : X \to Z$ such that:*

1. *f maps X and Y definably G homeomorphically onto G invariant definable subsets Z_1 and W_1 of Z and W obtained by removing some open G cells from Z and W, respectively.*
2. *The orbit map $\pi : Z \to Z/G$ is a definable cellular map.*
3. *The orbit space Z/G is a finite simplicial complex compatible with $\pi(Z_1)$ and $\pi(W_1)$.*
4. *For each open G cell c of Z, $\pi|\bar{c} : \bar{c} \to \pi(\bar{c})$ has a definable section $s : \pi(\bar{c}) \to \bar{c}$, where $\bar{c}$ denotes the closure of c in Z.*

Furthermore, if X is compact, then $Z = f(X)$ and $W = f(Y)$.

As applications of Theorem 1.1, we have the following three results.

Let X and Y be definable G sets. Two definable G maps $f, h : X \to Y$ *are definably G homotopic* if there exists a definable G map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = h(x)$ for all $x \in X$, where the action on $[0,1]$ of G is trivial. Let $[X,Y]^G_{def}$ (resp. $[X,Y]^G_{top}$) denote the set of definable G homotopy (resp. G homotopy) classes of definable G maps (resp. continuous G maps) from X to Y. Then we have a canonical map $\mu : [X,Y]^G_{def} \to [X,Y]^G_{top}, \mu([f]^G_{def}) = [f]^G_{top}$, where $[f]^G_{def}$ (resp. $[f]^G_{top}$) denotes the definable G homotopy (resp. G homotopy) class of f.

Theorem 1.2. *Let G be a compact definable group, and X and Y definable G sets. Then $\mu : [X,Y]^G_{def} \to [X,Y]^G_{top}, \mu([f]^G_{def}) = [f]^G_{top}$ is bijective.*

Let G be a finite group. A definable G vector bundle η over a definable G set X is *strongly definable* if there exist a representation Ω of G and a definable G map $f : X \to G(\Omega, k)$ such that η is definably G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η and $G(\Omega, k)$ means the universal G vector bundle associated with Ω and k (see Definition 4.4).

Let X be a definable G set. Let $Vect^G_{def}(X)$ (resp. $Vect^G_{top}(X)$) denote the set of definable G vector bundle (resp. G vector bundle) isomorphism classes of definable G vector bundles (resp. G vector bundles) over X. Then there is a canonical map $\kappa : Vect^G_{def}(X) \to Vect^G_{top}(X), \kappa([\eta]^G_{def}) = [\eta]^G_{top}$, where $[\eta]^G_{def}$ (resp. $[\eta]^G_{top}$) denotes the definable G vector bundle (resp. G vector bundle) isomorphism class of η.

Theorem 1.3. *Let G be a finite group and X a definable G set.*

(1) Every definable G vector bundle over X is strongly definable.

(2) The canonical map $\kappa : Vect^G_{def}(X) \to Vect^G_{top}(X), \kappa([\eta]^G_{def}) = [\eta]^G_{top}$ is bijective.

Let $1 \le r \le \omega$. Definable C^rG manifolds (resp. Definable C^rG vector bundles) are introduced in [12] (resp. [11]). For two definable C^rG manifolds X and Y, $\mu' : [X,Y]^G_{def\ C^r} \to [X,Y]^G_{top}, \mu'([f]^G_{def\ C^r}) = [f]^G_{top}$ and $\kappa' : Vect^G_{def\ C^r}(X) \to Vect^G_{top}(X)$, $\kappa'([\eta]^G_{def\ C^r}) = [\eta]^G_{top}$ are defined similarly.

The following is a definable C^rG version of Theorem 1.2 and 1.3. Recall that an *affine definable C^rG manifold* means a definable C^rG manifold which is definably C^rG diffeomorphic to a G invariant definable C^r submanifold of some representation of G.

Theorem 1.4. *Let G be a finite group, X, Y affine definable C^rG manifolds and $1 \leq r < \infty$.*

(1) The canonical map $\mu' : [X, Y]^G_{def\ C^r} \to [X, Y]^G_{top}$ is bijective.

(2) The canonical map $\kappa' : Vect^G_{def\ C^r}(X) \to Vect^G_{top}(X)$ is bijective.

2. Definable G sets and proof of Theorem 1.1

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets. A definable map $f : X \to Y$ is called a *definable homeomorphism* if there exists a definable map $h : Y \to X$ such that $f \circ h = id$ and $h \circ f = id$.

Theorem 2.1. *(1) (Definable triangulation (e.g. (8.2.9 [5])). Let $S \subset \mathbb{R}^n$ be a definable set and $S_1, \dots, S_k$ definable subsets of S. Then there exist a finite simplicial complex K in $\mathbb{R}^n$ and a definable map $\phi : S \to \mathbb{R}^n$ such that ϕ maps S and each S_i definably homeomorphically onto a union of open simplexes of K. If S is compact, then we can take $K = \phi(S)$.*
(2) (Piecewise definable trivialization (e.g. 9.1.2 [5])). Let X and Y be definable sets and $f : X \to Y$ a definable map. Then there exist a finite partition $\{T_i\}_{i=1}^k$ of Y into definable sets and definable homeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(y_i)$ such that $f|f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \leq i \leq k)$, where $y_i \in T_i$ and $p_i : T_i \times f^{-1}(y_i) \to T_i$ denotes the projection.

Let G and G' be definable groups. A *definable group homomorphism* $G \to G'$ means a group homomorphism which is a definable map. An *n-dimensional representation* of a definable group G means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from G to $O_n(\mathbb{R})$. A subgroup of a definable group is a *definable subgroup* of it if it is a definable subset of it. A definable map (resp. A definable homeomorphism) between definable G sets is a *definable G map* (resp. a *definable G homeomorphism*) if it is a G map.

Let G be a definable group. A *definable set with a definable G action* is a pair (X, θ) consisting of a definable set X and a group action $\theta : G \times X \to X$ such that θ is a definable map. This action is not necessarily linear (orthogonal). Similarly, we can define *definable G maps* and *definable G homeomorphisms* between them.

By [16], if H is a definable subgroup of a compact definable group G, then G/H is a definable set, and the standard action $G \times G/H \to G/H$ defined by $(g, g'H) \mapsto gg'H$ of G on G/H makes G/H a definable set with a definable G action. Furthermore every definable subgroup of a definable group is closed [17].

Definition 2.2. Let G be a compact definable group.

(1) A *definable G CW complex* is a finite G CW complex $(X, \{c_i | i \in I\})$ such that:

(a) The underlying space $|X|$ of X is a definable G set.

(b) The characteristic map $f_{c_i} : G/H_{c_i} \times \Delta \to \overline{c_i}$ of each open G cell c_i is a definable G map and $f_{c_i}|G/H_{c_i} \times \mathrm{Int}\,\Delta : G/H_{c_i} \times \mathrm{Int}\,\Delta \to c_i$ is a definable G homeomorphism, where H_{c_i} is a definable subgroup of G, Δ denotes a closed simplex, $\overline{c_i}$ is the closure of c_i in X, and Int Δ means the interior of Δ.

(2) Let X and Y be definable G CW complexes. A cellular G map $f : X \to Y$ is *definable* if $f : |X| \to |Y|$ is definable.

For the proof of Theorem 1.1, recall an equivariant version of Theorem 2.1 (2) proved in [11].

Theorem 2.3 (2.5 [11]). *Let G be a compact definable group, X a definable G set, Y a definable set, and $f : X \to Y$ a G invariant definable map. Then there exist a finite decomposition $\{T_i\}_{i=1}^k$ of Y into definable sets and definable G homeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(y_i)$ such that $f|f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \le i \le k)$, where p_i denotes the projection $T_i \times f^{-1}(y_i) \to T_i$ and $y_i \in T_i$.*

Proof of Theorem 1.1. Let Ω be a representation of G containing X as a G invariant definable subset and $\phi : \Omega \to \Omega$ a definable G homeomorphism $x \mapsto x/(1+||x||)$, where $||x||$ denotes the standard norm of x. Replacing X by $\phi(X)$, we may suppose that X is bounded. Then the closure $\overline{X}$ of X in Ω is a compact definable G set. By 10.2.8 [5], $\overline{X}/G$ is a compact definable set and the orbit map $\pi_{\overline{X}} : \overline{X} \to \overline{X}/G$ is a definable map.

By Theorem 2.3, there exist a finite decomposition $\{B_i\}_{i=1}^k$ of $\overline{X}/G$ into definable sets and definable G homeomorphisms $\phi_i : B_i \times \pi_{\overline{X}}^{-1}(b_i) \to \pi_{\overline{X}}^{-1}(B_i)$, $(1 \le i \le k)$, such that $\pi_{\overline{X}}|\pi_{\overline{X}}^{-1}(B_i) = p_i \circ \phi_i^{-1}$, $(1 \le i \le k)$, where $b_i \in B_i$ and p_i denotes the projection $B_i \times \pi_{\overline{X}}^{-1}(b_i) \to B_i$. By Theorem 2.1 and since $\overline{X}/G$ is compact, there exist a finite simplicial complex K and a definable homeomorphism $\tau : \overline{X}/G \to K$ such that τ maps each of $\pi_{\overline{X}}(X), \{B_i\}, \pi_{\overline{X}}(Y), cl(\pi_{\overline{X}}(Y))$ onto a union of open simplexes of K, where $cl(\pi_{\overline{X}}(Y))$ denotes the closure of $\pi_{\overline{X}}(Y)$ in $\overline{X}/G$. Note that $\tau(cl(\pi_{\overline{X}}(Y)))$ is a subcomplex of K.

We claim that each closed simplex $\Delta \in K$ admits a definable section $s : \tau^{-1}(\Delta) \to \pi_{\overline{X}}^{-1}(\tau^{-1}(\Delta))$ of $\pi_{\overline{X}}|\pi_{\overline{X}}^{-1}(\tau^{-1}(\Delta))$.

By the choice of a definable triangulation of $\overline{X}/G$, for each open simplex Int Δ, there exists a definable G homeomorphism $h : \pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta)) \to \pi_{\overline{X}}^{-1}(a) \times \tau^{-1}(\text{Int } \Delta)$ such that $\pi_{\overline{X}}|\pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta)) = p' \circ h$, where $p' : \pi^{-1}(a) \times \tau^{-1}(\text{Int } \Delta) \to \tau^{-1}(\text{Int } \Delta)$ denotes the projection onto the second factor and $a \in \tau^{-1}(\text{Int } \Delta)$. Thus we have a definable section $\tilde{s}$ of $\pi_{\overline{X}}|\pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta))$ defined by $\tilde{s}(x) = h^{-1}(b, x)$, where $b \in \pi_{\overline{X}}^{-1}(a)$. Since $\overline{X}$ is compact, Δ is a closed simplex and h is definable, we have a definable extension $s : \tau^{-1}(\Delta) \to \pi_{\overline{X}}^{-1}(\tau^{-1}(\Delta))$ of $\tilde{s}$. Thus the proof of the claim is complete.

Set $\sigma = s(\tau^{-1}(\Delta))$. Then $s \circ \tau^{-1} : \Delta \to \sigma$ is a definable homeomorphism. Hence there exists a definable G map $f_\sigma : G/H \times \Delta \cong G(b) \times \Delta \to G\sigma, (gH, x) \mapsto g(s\tau^{-1}(x)))$ such that $f_\sigma|G/H \times \text{Int } \Delta : G/H \times \text{Int } \Delta \to G\sigma$ is a definable G homeomorphism, where H denotes the isotropy subgroup of b. Furthermore f_σ itself is a definable G homeomorphism.

By collecting G cells $G\sigma = \pi_{\overline{X}}^{-1}(\tau^{-1}(\Delta))$ for all closed simplexes Δ of K, we have a definable G CW complex Z such that $|Z| = \overline{X}$ and $Z/G = \overline{X}/G$. Similarly, we have a subcomplex W of Z such that $|W| = \overline{Y}$ and $W/G = \overline{Y}/G$, where $\overline{Y}$ denotes the closure of Y in Ω. By the construction of Z, the orbit map $\pi : Z \to Z/G$ is a definable cellular map. Taking $Z_1 = \cup\{\pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta))|\Delta \in K, \tau^{-1}(\text{Int } \Delta) \subset \pi_{\overline{X}}(X)\}$ and $W_1 = \cup\{\pi_{\overline{X}}^{-1}(\tau^{-1}(\text{Int } \Delta))|\Delta \in K, \tau^{-1}(\text{Int } \Delta) \subset \pi_{\overline{X}}(Y)\}$, we have the required definable G homeomorphism f from (X, Y) to (Z_1, W_1). □

Remark that in the proof of Theorem 1.1, replacing K by any subdivision K^* of K, we have the corresponding subdivision of Z^* of Z instead of Z.

3. Proof of Theorem 1.2

Let X be a definable G set and Y a definable G subset of X. A *definable G retraction from X to Y* is a definable G map $r : X \to Y$ with $r|Y = id_Y$. A *definable strong G deformation retraction from X to Y* means a definable G map $R : X \times [0,1] \to X$ such that $R(x,0) = x$ for all $x \in X$, $R(y,t) = y$ for all $y \in Y, t \in [0,1]$ and $R(X,1) = Y$, where the action on $[0,1]$ is trivial. Note that $R(\cdot,1) : X \to Y$ is a definable G retraction from X to Y.

Let Z be a finite simplicial complex in $\mathbb{R}^n$ and X a union of open simplexes of Z. A subset Y of X is called a *subcomplex* of X if there exists a subcomplex Z_1 of Z with $Y = X \cap Z_1$. Note that every subcomplex of X is closed in X. The *first barycentric subdivision X' of X* is the intersection of the first barycentric subdivision Z' of Z with X. Similarly, the nth barycentric subdivision of X is defined. The *star $St_X(Y)$* (resp. *$St_{X'}(Y)$) of Y in X* (resp. X') is the union of all open simplexes σ of X (resp. X') with $\overline{\sigma} \cap Y \neq \emptyset$, where $\overline{\sigma}$ denotes the closure of σ in Z.

The above terms are defined similarly for definable G CW complexes.

Proposition 3.1 (2.2 [4]). *Let X be a union of open simplexes of a finite simplicial complex and Y a subcomplex of X. Then there exists a semialgebraic strong deformation retraction from the star $St_{X'}(Y)$ of Y in the first barycentric subdivision X' of X to Y.*

Remark that in Proposition 3.1 we cannot replace $St_{X'}(Y)$ by $St_X(Y)$.

Let X be a union of open simplexes of a finite simplicial complex Z. Then the maximal compact subcomplex Y of X' is $\{\sigma \in Z' | \overline{\sigma} \subset X'\}$ and $X' = St_{X'}(Y)$, where X' and Z' mean the first barycentric subdivisions of X and Z, respectively, and $\overline{\sigma}$ denotes the closure of σ in Z. Thus we the following corollary.

Corollary 3.2. *Let X be a union of open simplexes of a finite simplicial complex. Then X admits a semialgebraic strong deformation retraction from X to a compact semialgebraic subset Y of X.*

The following is the equivariant definable version of it.

Theorem 3.3. *Let G be a compact definable group and X a definable G set. Then there exists a definable strong G deformation retraction R from X to a compact definable G subset Y of X.*

Proof. Let Ω be a representation of G containing X as a definable G set. Then by Theorem 1.1, X is definably G homeomorphic to a union of open G cells of a definable G CW complex C in Ω. We identify X with its definably G homeomorphic image and replace C and X by their second barycentric subdivisions. For simplicity, we use the same letters C and X to mean them.

Let $f_c : G/H \times \Delta \to \overline{c} \subset C$ be the definable characteristic map of an open G cell c of X and put $\sigma = f_c(\{eH\} \times \text{Int } \Delta)$, where $\overline{c}$ denotes the closure of c in C. Note that $c = G\sigma$ and $\overline{c} = G\overline{\sigma} = \overline{G\sigma}$, where $\overline{\sigma}$ denotes the closure of σ in C.

Let Y denote the maximum compact G CW subcomplex of X. In other words, Y is the union of all open G cells c of X such that $\overline{c} \subset X$. Then $\overline{c} \cap Y \neq \emptyset$ for all open G cells c of X, thus the star $St_X(Y)$ of Y in X is X.

Let C_n be the set of open G n-cells c of X such that $c \cap Y = \emptyset$. Clearly each C_n is a finite set and $C_0 = \emptyset$. Let $X_0 = Y$ and $X_n = Y \cup X^{(n)}$ for $n \geq 1$, where $X^{(n)}$ denotes the union of open G r-cells c of X with $r \leq n$. Clearly $X_n = Y \cup \cup_{c \in \cup_{k=0}^n C_k} c$.

By the construction of a definable G CW complex structure C of X, for each open G n-cell $c \in C_n$, there exists a proper subset Δ' of Δ obtained by removing some lower dimensional faces of Δ such that $f_c^{-1}(\bar{c} \cap X) = G/H \times \Delta'$. Note that if $\bar{c} \subset X$, then $\bar{c} \subset Y$ by the construction of Y. Let $\delta = f_c(\{eH\} \times \Delta')$. Then $\sigma \subset \delta \subsetneq \bar{\sigma} = f_c(\{eH\} \times \Delta)$, cl $\sigma = \delta$ and $G\delta =$ cl c, where cl σ (resp. cl c) denotes the closure of σ (resp. c) in X.

Remark that there exists a semialgebraic strong deformation retraction $\Delta' \times [0,1] \to \Delta'$ from Δ' to $\partial \Delta' := \Delta' - \mathrm{Int}\ \Delta'$. Thus for each open G n-cell $c = G\sigma \in C_n$, there exists a definable strong H deformation retraction $F_\delta^n : \delta \times [0,1] \to \delta$ from δ to $\partial \delta := \delta - \mathrm{Int}\ \delta$, because the action H action on δ is trivial. Note that such a retraction exists because C and X are replaced by their second barycentric subdivisions. Using F_δ^n, we have a definable strong G deformation retraction

$$R_{G\delta}^n := G \times_H F_\delta^n : (G \times_H \delta) \times [0,1] \to G \times_H \delta$$

from $G \times_H \delta$ to $G \times_H \partial\delta$. Since $G \times_H \delta \cong G\delta$ and $G \times_H \partial\delta \cong G\partial\delta$, it gives a definable strong G deformation retraction from $G\delta$ to $G\partial\delta$ ($\subset X_{n-1}$).

Hence $\cup\{R_{G\delta}^n | c \in C_n\}$ induces a definable strong G deformation retraction $R^n : X_n \times [0,1] \to X_n$ from X_n to X_{n-1}. We can define $R^{n-1} \bullet R^n : X_n \times [0,1] \to X_n$,

$$R^{n-1} \bullet R^n(x,t) = \begin{cases} R^n(x,2t) & \text{if } 0 \leq t \leq 1/2 \\ R^{n-1}(R^n(x,1), 2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Thus the required definable strong G deformation retraction $R = R^1 \bullet R^2 \bullet \cdots \bullet R^{m-1} \bullet R^m : X \times [0,1] \to X$ from X to Y is obtained inductively, where $m = \min\{n \in \mathbb{N} | X = X_n\}$. □

The following is useful to prove our results.

Theorem 3.4. *Let G be a compact definable group and Y a closed definable G subset of a definable G set X. Then there exists a G invariant definable open neighborhood U of Y in X such that Y is a definable strong G deformation retract of both U and of the closure cl U of U in X.*

Proof. By Theorem 1.1, we may assume that X is a union of open G cells of a definable G CW complex C. We replace C and X by their second barycentric subdivisions. We use the same notations as in the proof of Theorem 3.3 unless otherwise specified.

Let $U = St_X(Y)$. Let S_n be the set of open G n-cells c of $St_X(Y)$ such that $c \cap Y = \emptyset$, and put $X_0 = Y$ and $X_n = Y \cup \cup_{c \in \cup_{k=0}^n S_k} c$. Note that cl $U = \cup\{$cl $c | c$ is an open G cell of $U\}$, where cl c denotes the closure of c in X.

Let $f_c : G/H \times \Delta \to \bar{c} \subset C$ be the definable characteristic map of an open G cell $c \in S_n$, where $\bar{c}$ denotes the closure of c in C. As in the proof of Theorem 3.3, we can find a subset Δ' of Δ obtained by removing some lower dimensional faces of Δ such that $f_c^{-1}(\bar{c} \cap X) = G/H \times \Delta'$. Thus there exists a semialgebraic strong deformation retraction from Δ' to the union Δ'' of faces d of $\partial\Delta' := \Delta' - \mathrm{Int}\ \Delta'$ such that $d' \cap \overline{f_c}^{-1}(Y) \neq \emptyset$, where d' denotes the closure of d in Δ' and $\overline{f_c} : \Delta' \to X$ is the composition of $\Delta' \to \{eH\} \times \Delta', x \mapsto (eH, x)$ with f_c. Note that Δ'' is a proper subset of $\partial\Delta'$ and such a retraction exists because C and X are replaced by their second barycentric subdivisions. As in the proof of Theorem

3.3, using this retraction, we have a definable strong G deformation retraction $R^n_{G\delta}$ from $G\delta$ to $G\tilde{\delta} \subset S_{n-1}$, where $\tilde{\delta}$ denotes the union of faces e of $\partial\delta$ such that the closure of e in X intersects with Y. Hence $\cup\{R^n_{G\delta}|c \in S_n\}$ induces a definable strong G deformation retraction from S_n to S_{n-1}. Thus, as in the proof of Theorem 3.3, we have the required definable strong G deformation retraction from both U and cl U to Y. □

The following proposition shows the surjectivity of μ in Theorem 1.2.

Proposition 3.5. *Let G be a compact definable group, X and Y be definable G sets. Then every continuous G map $f : X \to Y$ is G homotopic to a definable G map.*

To prove Proposition 3.5, we need the following lemma. It is proved by the polynomial approximation theorem and an observation similar to 4.3 [11].

Lemma 3.6. *Let G be a compact definable group and X a compact definable G set. Then every continuous G map f from X to a representation Ω of G is approximated by polynomial G maps.*

Proof of Proposition 3.5. Let Y be a definable G set in a representation Ξ of G. By Theorem 3.3, there exists a definable strong G deformation retraction $R_Y : Y \times [0,1] \to Y$ from Y to a compact definable G subset B of Y. Put $K : X \times [0,1] \to Y, K(x,t) = R_Y(f(x),t)$. Then K is a G homotopy from f to $r_Y \circ f$, where $r_Y := R_Y(\cdot,1)$.

Assume that X is compact. We now construct a definable G map which is G homotopic to $r_Y \circ f$. Since B is compact, there exists a real number $r > 0$ such that B is contained in the interior of $D := \{x \in \Xi | ||x|| \le r\}$. By Theorem 1.1, we may assume that (D, B) is a pair of a definable G CW complex and its G CW subcomplex. By Theorem 3.4, there exists a definable G retraction r_V from a G invariant definable open neighborhood V of B in D to B. By Lemma 3.6, we can approximate $r_Y \circ f$ by a polynomial G map $p : X \to \Xi$. Since V is an open subset of Ξ, we can take p with $p(X) \subset V$ if p is sufficiently close to $r_Y \circ f$. We may assume the line segment $(1-t)(r_Y \circ f)(x) + tp(x),\ 0 \le t \le 1$, lies in V. Then $h := r_V \circ p : X \to B$ is a definable G approximation of $r_Y \circ f$.

The map $P : X \times [0,1] \to B$ defined by $P(x,t) = r_V((1-t)(r_Y \circ f)(x) + tp(x))$ is a G homotopy from $r_Y \circ f$ to h. Thus the homotopy composition $K * P : X \times [0,1] \to Y$,

$$K * P(x,t) = \begin{cases} K(x,2t) & \text{if } 0 \le t \le 1/2 \\ P(x,2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

is a G homotopy from f to h. Therefore the result follows in this case.

Now assume that X is general. By Theorem 3.3, we can find a definable strong G deformation retraction $R_X : X \times [0,1] \to X$ from X to a compact definable G subset A of X. By the compact case, there exist a G homotopy $F : A \times [0,1] \to Y$ and a definable G map $u : A \to Y$ such that $F(x,0) = f(x), F(x,1) = u(x)$ for all $x \in A$. Put $H = F \circ (r_X \times id_{[0,1]}) : X \times [0,1] \to Y$, where $r_X := R_X(\cdot,1)$. Then H is a G homotopy from $f \circ r_X$ to $u \circ r_X$. Note that $f = f \circ id_X \underset{f \circ R_X}{\sim} f \circ r_X \underset{H}{\sim} u \circ r_X$. Therefore f is G homotopic to a definable G map $h := u \circ r_X$. □

A pair (X, Y) consisting of a definable G set X and a definable G subset Y of X *admits a definable G homotopy extension* if for any definable G map f from X to a definable G set Z and any definable G homotopy $F : Y \times [0,1] \to Z$ with $F(y,0) = f(y)$ for all

$y \in Y$, there exists a definable G homotopy $H : X \times [0,1] \to Z$ such that $H(x,0) = f(x)$ for all $x \in X$ and $H|Y \times [0,1] = F$.

Theorem 3.7. *Let G be a compact definable group. If X is a definable G set and Y is a closed definable G subset of X, then $Y \times [0,1] \cup (X \times \{0\})$ is a definable strong G deformation retract of $X \times [0,1]$. In particular (X,Y) admits a definable G homotopy extension.*

To prove Theorem 3.7, we need the following result.

Proposition 3.8. *Let G be a compact definable group and A, B disjoint definable closed G subsets of a definable G set X. Then there exists a G invariant definable map $f : X \to [0,1]$ with $A = f^{-1}(0)$ and $B = f^{-1}(1)$.*

Proof. By 10.2.8 [5], X/G is a definable set and the orbit map $\pi : X \to X/G$ is a definable map. Since π is closed and by 6.3.8 [5], there exists a definable map $h : X/G \to \mathbb{R}$ with $\pi(A) = h^{-1}(0)$ and $\pi(B) = h^{-1}(1)$. Thus $f := h \circ \pi : X \to \mathbb{R}$ is the required G invariant definable map. $\square$

Proof of Theorem 3.7. By Theorem 3.4, there exist a G invariant definable open neighborhood U of Y in X and a definable strong G deformation retraction $H : \mathrm{cl}\, U \times [0,1] \to \mathrm{cl}\, U$ from $\mathrm{cl}\, U$ to Y, where $\mathrm{cl}\, U$ denotes the closure of U in X. By Proposition 3.8, we have a G invariant definable map $\lambda : X \to [0,1]$ with $\lambda^{-1}(0) = X - U$ and $\lambda^{-1}(1) = Y$. Put

$$B = \{(x,t) \in \mathrm{cl}\, U \times [0,1] | \tfrac{1}{2} \leq \lambda(x) < 1, 2(1-\lambda(x)) \leq t \leq 1\},$$
$$C = \{(x,t) \in \mathrm{cl}\, U \times [0,1] | \tfrac{1}{2} \leq \lambda(x) < 1, 0 \leq t \leq 2(1-\lambda(x))\},$$
$$D = \{(x,t) \in \mathrm{cl}\, U \times [0,1] | 0 \leq \lambda(x) \leq \tfrac{1}{2}\}, \text{and } E = (X - U) \times [0,1].$$

Then B, C, D, E are G invariant definable subsets of $X \times [0,1]$ such that $X \times [0,1] = (Y \times [0,1]) \cup B \cup C \cup D \cup E$, D and E are closed in $X \times [0,1]$, $B' = B \cup (Y \times [0,1])$ and that $C' = C \cup (Y \times \{0\})$, where B' (resp. C') denotes the closure of B (resp. C) in $X \times [0,1]$. Define $\psi : C \to [0,1], \psi(x,t) = \frac{t}{2(1-\lambda(x))}$. Then ψ is a G invariant definable function. Now we define a definable G retraction $R : X \times [0,1] \to (Y \times [0,1]) \cup (X \times \{0\})$,

$$R(x,t) = \begin{cases} (r(x), t - 2(1-\lambda(x)) & \text{if } (x,t) \in B \cup (Y \times [0,1]) \\ (H(x, \psi(x,t)), 0) & \text{if } (x,t) \in C \\ (H(x, 2t\lambda(x)), 0) & \text{if } (x,t) \in D \\ (x, 0) & \text{if } (x,t) \in E \end{cases},$$

where $r := H(\cdot, 1)$. Then R is a well-defined definable map.

To see continuity of R, it suffices to check that for a given point y of Y, $R(x,t)$ converges to $(y,0)$ if $(x,t) \in C$ and (x,t) tends to $(y,0)$. Since H is continuous at (y,t), for any $\epsilon > 0$, there exists $\delta' > 0$ such that $||x-y|| < \delta', |t'-t| < \delta' \Rightarrow ||H(x,t') - H(y,t)|| < \epsilon$, where $||z||$ denotes the standard norm of z in a representation of G containing X. By compactness of $[0,1]$, there exists $\delta > 0$ such that $||x-y|| < \delta \Rightarrow ||H(x,t) - y|| = ||H(x,t) - H(y,t)|| < \epsilon$ for any $t \in [0,1]$. Thus $R(x,t) \to (y,0)$ as $(x,t) \to (y,0)$. Notice that $\lim_{(x,t)\to(y,0),(x,t)\in C} \psi(x,t)$ does not necessarily exist.

Since the path $H(x,t)$ from x to $r(x)$ is contained in $\mathrm{cl}\, U$ for any $x \in \mathrm{cl}\, U$, we can define a definable G map $\Psi : (X \times [0,1]) \times [0,1] \to X \times [0,1]$,

$$\Psi(x,t,s) = \begin{cases} (H(x,s), t-2s(1-\lambda(x)) & \text{if } (x,t) \in B \cup (Y \times [0,1]) \\ (H(x, s\psi(x,t)), t(1-s)) & \text{if } (x,t) \in C \\ (H(x, 2st\lambda(x)), t(1-s)) & \text{if } (x,t) \in D \\ (x, t(1-s)) & \text{if } (x,t) \in E \end{cases}.$$

Then Ψ has a definable graph. The continuity of Ψ is checked similarly. Therefore Ψ is the required definable strong G deformation retraction from $X \times [0,1]$ to $(Y \times [0,1]) \cup X \times \{0\}$ such that $\Psi(x,t,0) = (x,t)$ and $\Psi(x,t,1) = R(x,t)$ for any $(x,t) \in X \times [0,1]$. □

To prove Theorem 1.2, we need a relative version of Proposition 3.5.

Let X and Y be definable G sets, C a definable G subset of X, and $\phi : C \to Y$ a definable G map. We say that two definable G extensions $f, h : X \to Y$ of ϕ *are definably G homotopic relative to C* if there exists a definable G map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x), H(x,1) = h(x)$ for all $x \in X$ and $H(c,t) = \phi(c)$ for all $(c,t) \in C \times [0,1]$. Let $[X,Y]_{def}^{G,\phi}$ (resp. $[X,Y]_{top}^{G,\phi}$) denote the set of definable G homotopy (resp. G homotopy) classes of definable G maps (resp. continuous G maps) from X to Y extending ϕ relative to C. Then we have a canonical map $\tilde{\mu} : [X,Y]_{def}^{G,\phi} \to [X,Y]_{top}^{G,\phi}, \tilde{\mu}([f]_{def}^{G,\phi}) = [f]_{top}^{G,\phi}$, where $[f]_{def}^{G,\phi}$ (resp. $[f]_{top}^{G,\phi}$) denotes the definable G homotopy (resp. G homotopy) class of f relative to C.

Proposition 3.9. *Let G be a compact definable group, X, Y definable G sets, and C a definable closed G subset of X. Then for a given definable G map $\phi : C \to Y$, $\tilde{\mu} : [X,Y]_{def}^{G,\phi} \to [X,Y]_{top}^{G,\phi}$ is surjective.*

Proof. By Theorem 1.1, we may assume that X is a union of open G cells of a definable G CW complex and that C is a subcomplex of X. We replace them by their second barycentric subdivisions, and use the same letters.

Let $f : X \to Y$ be a continuous G map with $f|C = \phi$. By Theorem 3.4, there exists a definable strong G deformation retraction R from a G invariant definable closed neighborhood D of C in X to C. Let $r = R(\cdot, 1)$. Then there exist a definable G map $\phi_1 := \phi \circ r : D \to Y$ and a G homotopy $L : D \times [0,1] \to Y$ from $f|D$ to ϕ_1 such that $L(c,t) = \phi(c)$ for all $(c,t) \in C \times [0,1]$. By Theorem 3.7, $D \times [0,1] \cup (X \times \{0\})$ is a definable strong G deformation retract of $X \times [0,1]$. Thus L is extendable to a G homotopy $F : X \times [0,1] \to Y$ with $F(x,0) = f(x)$ for all $x \in X$ and $F|D \times [0,1] = L$. Let $f_1 = F(\cdot, 1)$. Then $f_1|D = \phi_1$.

By Theorem 3.3, we can find a definable strong G deformation retraction $R_X : X \times [0,1] \to X$ (resp. $R_Y : Y \times [0,1] \to Y$) from X (resp. Y) to a compact definable G subset X_1 (resp. Y_1) of X (resp. Y). Using X_1, Y_1, we have a G homotopy $H : X \times [0,1] \to Y$ from f_1 to a definable G map f_2 as in the proof of Proposition 3.5. By the construction of H and since $f_1|D$ $(= \phi_1)$ is a definable G map, $H|D \times [0,1]$ is a definable G map. However H does not necessarily satisfy the condition that $H(c,t) = \phi(c)$ for all $(c,t) \in C \times [0,1]$. By Proposition 3.8, there exists a G invariant definable map $\lambda : X \to [0,1]$ with $\lambda^{-1}(0) = C$ and $\lambda^{-1}(1) = X - \mathrm{Int}\, D$. Define a G homotopy $\tilde{H} : X \times [0,1] \to Y, \tilde{H}(x,t) = H(x, t\lambda(x))$. Then $\tilde{H}(c,t) = f_1(c) = \phi(c)$ for all $(c,t) \in C \times [0,1], \tilde{H}(x,0) = f_1(x)$ for all $x \in X$, and $h(x) := \tilde{H}(x,1) = H(x, \lambda(x))$ is a definable G map because $h|X - D = f_2|X - D$ and

$H|D \times [0,1]$ is a definable G map. Therefore f is G homotopic to h and its homotopy is provided by the homotopy composition $F * \tilde{H}$ of F with $\tilde{H}$. $\square$

Proof of Theorem 1.2. It suffices to prove the injectivity of μ. Let $f, h : X \to Y$ be two definable G maps and $F : X \times [0,1] \to Y$ a G homotopy between f and h. Since $C := X \times \{0,1\}$ is closed in $X \times [0,1]$ and $\phi : C \to Y, \phi := f \amalg h$ is a definable G map and by Proposition 3.9, there exists a G homotopy between F and a definable G map $F' : X \times [0,1] \to Y$ relative to C. Therefore F' is a required definable G homotopy between f and h. $\square$

4. Proof of Theorem 1.3 and 1.4

As a generalization of a semialgebraic space, we can consider a *definable space* which is a topological space obtained by gluing finitely many definable sets with definable homeomorphisms (see section 10.1 [5]). Clearly a definable set is a definable space. Similarly, we can define a *definable map* between definable spaces (see section 10.1 [5]).

Let G be a definable group. A *definable G space* is a pair (X, θ) consisting of a definable space X and a group action $\theta : G \times X \to X$ of G such that θ is a definable map. Note that a definable G set is a definable G space. A definable map between definable G spaces is a *definable G map* if it is a G map.

Definition 4.1. (1) Let $\eta = (E, p, X)$ be a vector bundle of rank k over a definable set X. A finite family of local trivializations $(U_i, \varphi_i \colon U_i \times \mathbb{R}^k \to p^{-1}(U_i))_{i \in I}$ of η is said to be a *definable atlas* of η if $(U_i)_{i \in I}$ is a finite definable open covering of X and for every pair $(i,j) \in I \times I$, the map $\varphi_i^{-1} \circ \varphi_j | (U_i \cap U_j) \times \mathbb{R}^k : (U_i \cap U_j) \times \mathbb{R}^k \to (U_i \cap U_j) \times \mathbb{R}^k$ is definable. Two definable atlases are *equivalent* if their union is still a definable atlas. A *definable vector bundle* is a vector bundle $\eta = (E, p, X)$ equipped with an equivalence class of definable atlases.

(2) Let $(\eta, (U_i, \varphi_i)_{i \in I})$ and $(\eta', (U'_j, \varphi'_j)_{j \in J})$ be two definable vector bundles over a definable set X. A vector bundle morphism $\psi : \eta \to \eta'$ is said to be a *definable vector bundle morphism* if for every $(i,j) \in I \times J$, the map $(\varphi'_j)^{-1} \circ \psi \circ \varphi_i | (U_i \cap U'_j) \times \mathbb{R}^k : (U_i \cap U'_j) \times \mathbb{R}^k \to (U_i \cap U'_j) \times \mathbb{R}^k$ is definable. A definable vector bundle morphism $h : \eta \to \eta'$ is a *definable vector bundle isomorphism* if there exists a definable vector bundle morphism $k : \eta' \to \eta$ such that $h \circ k = id$ and $k \circ h = id$. A continuous section s of η is said to be a *definable section* if for every $i \in I$, the map $\varphi_i^{-1} \circ s | U_i : U_i \to U_i \times \mathbb{R}^k$ is definable.

By abuse of notion, we denote by $\eta = (E, p, X)$ a definable vector bundle without specifying the atlas defining its structure. Note that the total space of a definable vector bundle is a definable space.

Definition 4.2. Let G be a definable group.

(1) A definable vector bundle $\eta = (E, p, X)$ is a *definable G vector bundle* if η satisfies the following two conditions:

(a) The total space E is a definable G space and the base space X is a definable G set.

(b) The projection $p \colon E \to X$ is a definable G map, and for any $x \in X$ and $g \in G$, $p^{-1}(x) \to p^{-1}(gx), y \mapsto gy$ is a linear isomorphism.

(2) A *definable G vector bundle morphism* $f : \eta \to \eta'$ between two definable G vector bundles $\eta = (E, p, X)$ and $\eta' = (E', p', X)$ is a definable G map $f : E \to E'$ such that

$p' \circ f = p$ and f is linear on each fiber. A definable G vector bundle morphism $h : \eta \to \eta'$ is called a *definable G vector bundle isomorphism* if there exists a definable G vector bundle morphism $k : \eta' \to \eta$ such that $h \circ k = id$ and $k \circ h = id$.

(3) A definable section s of a definable G vector bundle is called a *definable G section* if it is a G map.

By a way similar to 3.1 [10], we have the following.

Proposition 4.3. *Let G be a definable group. If η and η' are two definable G vector bundle over a definable G set X, then $\eta \oplus \eta', \eta \otimes \eta'$, $Hom(\eta, \eta')$ and the dual bundle $\eta^\vee$ of η are definable G vector bundles over X.*

Recall universal G vector bundles (e.g. [11]).

Definition 4.4. Let G be a finite group and $0 \leq r \leq \omega$. Let Ω be an n-dimensional representation of G and let B be the representation map $G \to O_n(\mathbb{R})$ of Ω. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \to B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer k, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, A = A', Tr A = k\},$$
$$E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\},$$
$$u : E(\Omega, k) \to G(\Omega, k) : u((A, v)) = A,$$

where A' denotes the transposed matrix of A and $Tr\ A$ stands for the trace of A. Then $\gamma(\Omega, k)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic G vector bundle. We call it *the universal G vector bundle associated with Ω and k*. Remark that $G(\Omega, k) \subset M(\Omega)$ and $E(\Omega, k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic G sets.

Proof of Theorem 1.3 (1). Let η be a definable G vector bundle over X. Then by a way similar to 12.7.4 [3], we can find a definable section $s_1, \dots, s_k$ of η such that the vectors $s_1(x), \dots, s_k(x)$ generate the fiber $p^{-1}(x)$ for all $x \in X$. Remember that the set $\Gamma(\eta)$ of continuous sections of η has a natural G action, namely $(g \cdot s)(x) = g(s(g^{-1}(x))), s \in \Gamma(\eta), g \in G$ and $x \in X$. Since G is finite, we have a finite family of definable sections $\{g \cdot s_i | 1 \leq i \leq k, g \in G\} \subset \Gamma(\eta)$ which is G invariant.

Hence this family of sections defines a representation Ω of G, and for each $x \in X$, $\{gs_i(x) | 1 \leq i \leq k, g \in G\}$ defines a vector subspace V_x of Ω. Therefore the orthogonal projection from Ω onto V_x induces a definable G map $F : X \to G(\Omega, k)$ such that η is definably G vector bundle isomorphic to $F^*(G(\Omega, k))$. □

Proposition 4.5 ([2], [13]). *Let G be a compact Lie group, X a paracompact G space, and η a G vector bundle over a G space Y. If $f, h : X \to Y$ are G homotopic continuous G maps, then $f^*(\eta)$ and $h^*(\eta)$ are G vector bundle isomorphic.*

Proposition 4.6 ([1], [18]). *Let G be a compact topological group and X a compact G space. If η is a G vector bundle, then there exist a representation Ω of G and a continuous G map $f : X \to G(\Omega, k)$ such that η is G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$.*

By Proposition 4.5, 4.6, 3.5 and Theorem 3.3, we have the surjectivity of κ.

Proposition 4.7. *Let G be a finite group and η a definable G vector bundle over a compact definable G set X. Then every continuous G section of η can be approximated by definable G sections.*

Proof. By Theorem 1.3 (1), η is strongly definable. Hence one can find a representation Ω of G and a definable G map $f : X \to G(\Omega, k)$ such that η is definably G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η. Thus we can identify η with a subbundle of the trivial G vector bundle $\underline{\Omega} = X \times \Omega$. Under this identification, a map $h : X \to \Omega$ is a section of η if and only if $f(x)h(x) = h(x)$ for any $x \in X$. Let l be a continuous G section of η. We regard l as a continuous G map X to Ω. By Lemma 3.6, there exists a polynomial G map $p : X \to \Omega$ as an approximation of l. Put $s(x) = f(x)p(x)$. Then we have $f(x)s(x) = f(x)^2p(x) = f(x)p(x) = s(x)$ for any $x \in X$ because $f(x) \in G(\Omega, k)$ for any $x \in X$. Therefore s is a definable G section approximating l. □

The following theorem proves the injectivity of κ when X is compact.

Theorem 4.8. *Let G be a finite group. Let η and ζ be definable G vector bundles over a compact definable G set. If η is G vector bundle isomorphic to ζ, then they are definably G vector bundle isomorphic.*

Proof. By Proposition 4.3 and Theorem 1.3 (1), Hom (η, ζ) is strongly definable. Take a G vector bundle isomorphism f between η and ζ. We can see f as a continuous G section of Hom (η, ζ) which lies in Iso (η, ζ). By Proposition 4.7, there exists a definable G section s of Hom (η, ζ) approximating f. If this approximation is sufficiently close, then s gives the required definable G vector bundle isomorphism because Iso (η, ζ) is open in Hom (η, ζ). □

Using Proposition 4.5, we have the following corollary.

Corollary 4.9. *Let G be a finite group, X a compact G contractible definable G set. Then every definable G vector bundle over X is definably G vector bundle isomorphic to a trivial G bundle.*

Let X be a definable G set. By Theorem 3.3, one can find a definable G retraction r from X to a compact definable G subset Y of X. Let $i : Y \to X$ denote the inclusion. Then $r^* : Vect^G_{def}(Y) \to Vect^G_{def}(X)$ is injective and $i^* : Vect^G_{def}(X) \to Vect^G_{def}(X)$ is surjective because $r \circ i = id_Y$.

Proposition 4.10. *Let X, Y, r be as in the immediately above paragraph. Then r^* : $Vect^G_{def}(Y) \to Vect^G_{def}(X)$ is bijective.*

Proof. By Theorem 1.1, we may assume that X is a union of open G cells of a definable G CW complex C. We replace X and C by their second barycentric subdivisions. We use the same notation as in the proof of Theorem 3.3.

Remember that the definable strong G deformation retraction R from X to Y constructed in Theorem 3.3 is $R^1 \bullet R^2 \bullet \cdots \bullet R^{m-1} \bullet R^m$. Note that $r_i := R_i(\cdot, 1)$ is a definable G retraction from X_i to X_{i-1} and the definable G retraction r from X to Y is given by $r := r_1 \circ \cdots \circ r_m$ $(= R(\cdot, 1))$. By construction, for each n with $1 \leq n \leq m$, $r|X_n : X_n \to Y$ $(= r_1 \circ \cdots \circ r_n)$ is a definable G retraction from X_n to Y.

Let η be a definable G vector bundle over X. By induction, we now construct a definable G vector bundle isomorphism $\Phi : \eta \to r^*(\eta|Y)$. Assume that we have a definable G vector bundle isomorphism $\Phi_{n-1} : \eta|X_{n-1} \to (r|X_{n-1})^*(\eta|Y)$. Then it induces a definable G vector bundle isomorphism $\Phi'_n : (r_n)^*(\eta|X_{n-1}) \to (r_n)^*(r|X_{n-1})^*(\eta|Y) \cong (r|X_n)^*(\eta|Y)$.

For an open G-n cell $c \in C_n$, let $f_c : G/H \times \Delta \to \bar{c} \subset C$ denote its definable characteristic map. As in the proof of Theorem 3.3, we can find a proper subset Δ' of Δ obtained by removing some lower dimensional faces of Δ such that $f_c^{-1}(\bar{c} \cap X) = G/H \times \Delta'$. Let $\delta = f_c(\{eH\} \times \Delta')$ and $\sigma = f_c(\{eH\} \times \mathrm{Int}\, \Delta)$. Then the H actions on $\{eH\} \times \Delta'$ and δ are trivial, and $r_n|G\delta : G\delta \to G\delta$ is a definable G retraction from $G\delta$ to $G\partial\delta$.

Put $\zeta = \overline{f_c}^*(\eta)$, where $\overline{f_c} : \Delta' \to X$ denotes the composition of $\Delta' \to \{eH\} \times \Delta', x \mapsto (eH, x)$ with f_c. By Theorem 1.3 (1), ζ is strongly definable. Thus we have a definable H map $\varphi : \Delta' \to G(\Omega, k)$ such that ζ is definably H vector bundle isomorphic to $\varphi^*(\gamma(\Omega, k))$, where k denotes the rank of η. Since $G(\Omega, k)$ is compact, Δ is a closed simplex and φ is definable, φ has a definable H extension $\varphi' : \Delta \to G(\Omega, k)$. Using φ', we get a strongly definable H vector bundle $(\varphi')^*(\gamma(\Omega, k))$ over Δ such that $(\varphi')^*(\gamma(\Omega, k))|\Delta'$ is definably H vector bundle isomorphic to ζ.

Since Δ is a compact H contractible definable H set and by Corollary 4.9, $(\varphi')^*(\gamma(\Omega, k))$ is definably H vector bundle isomorphic to a trivial definable H vector bundle $\Delta \times V$ for some representation V of H. In particular, ζ is trivial.

Remember that $F^n_\delta : \delta \times [0,1] \to \delta$ is a definable strong H deformation retraction from δ to $\partial\delta$. Let $r_\delta := F^n_\delta(\cdot, 1)$. Recall that the characteristic map $f_c : G/H \times \Delta \to \bar{c} \subset C$ is itself a definable G homeomorphism as in the proof of Theorem 1.1. Since ζ is trivial, so is $\eta' := \eta|\delta$. We identify η' with $\delta \times V$ and $\eta'|\partial\delta$ with $\partial\delta \times V$. Let $l : \eta'|\partial\delta \to \partial\delta \times V, l(x, v) = (x, l_x(v))$ be a definable H vector bundle isomorphism. Then the definable H vector bundle isomorphism $\delta \times V \to \delta \times V$ defined by $(x, v) \mapsto (x, l_{r_\delta(x)}(v))$ induces a definable H vector bundle isomorphism $\Psi_\delta : \eta' \to r^*_\delta(\eta'|\partial\delta)$ such that $\Psi_\delta|\partial\delta$ is the identity. Hence we have a definable G vector bundle isomorphism $G \times_H \Psi_\delta : G \times_H \eta' \to G \times_H (r^*_\delta(\eta'|\partial\delta))$ such that $G \times_H \Psi_\delta|G \times_H \partial\delta$ is the identity. It induces a definable G vector bundle isomorphism $\Psi_{G\delta} : \eta|G\delta \to (r_n|G\delta)^*(\eta|G\partial\delta)$ such that $\Psi_{G\delta}|G\partial\delta$ is the identity. Thus it provides a definable G vector bundle isomorphism $\Psi_n : \eta|X_n \to (r_n)^*(\eta|X_{n-1})$. Hence we have a definable G vector bundle isomorphism $\Phi_n : \eta|X_n \to (r|X_n)^*(\eta|Y)$ defined by $\Phi_n = \Phi'_n \circ \Psi_n$. Therefore $\Phi = \Phi_m$ is the required definable G vector bundle isomorphism and r^* is bijective. $\square$

Proof of Theorem 1.3 (2). It suffices to prove injectivity of κ. By Proposition 4.10 and 4.5, the induced maps $r^* : Vect^G_{def}(Y) \to Vect^G_{def}(X), r^* : Vect^G_{top}(Y) \to Vect^G_{top}(X)$ by $r : Y \to X$ are bijective. Let $\kappa_Y : Vect^G_{def}(Y) \to Vect^G_{top}(Y), \kappa_Y([\eta]^G_{def}) = [\eta]^G_{top}$. Then two maps $\kappa \circ r^*, r^* \circ \kappa_Y : Vect^G_{def}(Y) \to Vect^G_{top}(X)$ coincide. Since Y is compact, κ_Y is bijective. Therefore κ is bijective. Therefore the proof of Theorem 1.3 (2) is complete. $\square$

To prove Theorem 1.4, we need the following four results.

Theorem 4.11 (4.13 [11]). *Let G be a finite group. Let X and Y be affine definable C^rG manifolds and $0 \le k < r < \infty$. Then every definable C^kG map $f : X \to Y$ is approximated in the definable C^k topology by definable C^rG maps.*

Note that if X is compact, then the definable C^k topology coincides with the C^k Whitney topology. Detailed properties of the definable C^k topology can be seen in [11].

Lemma 4.12 (4.12 [11]). *If G is a finite group and $0 < r < \infty$, then for every definable C^rG submanifold X in a representation Ω of G, there exist a G invariant definable open neighborhood U of X in Ω and a definable C^rG map $p : U \to X$ such that $p|X = id_X$.*

The following proposition is an equivariant definable C^r version of Proposition 3.8.

Proposition 4.13. *Let G be a finite group, A and B disjoint definable closed G subsets of an affine definable C^rG manifold X and $0 \le r < \infty$. Then there exists a G invariant definable C^r function $f : X \to [0,1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.*

Proof. Using [8], there exists a nonequivariant definable C^r function $f_1 : X \to [0,1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. Then the averaged function $f : X \to [0,1]$ of f_1 defined by $f(x) = \frac{1}{n}\sum_{i=1}^n f_1(g_i x)$ is the required function, where $G = \{g_1, \dots, g_n\}$. □

Proposition 4.14 (1.8 [11]). *Let G be a finite group, X an affine definable C^rG manifold and $1 \le r < \infty$. Then for any two definable C^rG vector bundle over X, if they are definable G vector bundle isomorphic, then they are definably C^rG vector bundle isomorphic.*

Proof of Theorem 1.4. We first prove (1). Let $f : X \to Y$ be a continuous G map. Then by Theorem 1.2, f is G homotopic to a definable G map $f' : X \to Y$. By Theorem 4.11, we have a definable C^rG map $h : X \to Y$ as an approximation of f'. If this approximation is sufficiently close, then using Lemma 4.12, one can show that f' is G homotopic to h. Thus f and h are G homotopic. Therefore surjectivity of μ' is proved

Assume that two definable C^rG maps $f_1, f_2 : X \to Y$ are G homotopic. Then by Theorem 1.2, they are definably G homotopic. Take a definable G homotopy $X \times [0,1] \to Y$ from f_1 to f_2, and we extend it to a definable G map $H : X \times \mathbb{R} \to Y$. By Theorem 4.11, there exists a definable C^rG map $H' : X \times \mathbb{R} \to Y$ approximating H.

For any positive G invariant definable function $\epsilon : X \to \mathbb{R}$, $\{(x,t) \in X \times \mathbb{R} | t \le \epsilon(x)\}$ and $\{(x,t) \in X \times \mathbb{R} | t \ge 2\epsilon(x)\}$ are disjoint definable closed G subsets of $X \times \mathbb{R}$. Thus by Proposition 4.13, there exists a G invariant definable C^r function $\lambda_1 : X \times \mathbb{R} \to [0,1]$ such that $\lambda_1^{-1}(0) = \{(x,t) \in X \times \mathbb{R} | t \le \epsilon(x)\}$ and $\lambda_1^{-1}(1) = \{(x,t) \in X \times \mathbb{R} | t \ge 2\epsilon(x)\}$. Similarly, we have a G invariant definable C^r function $\lambda_2 : X \times \mathbb{R} \to [0,1]$ such that $\lambda_2^{-1}(1) = \{(x,t) \in X \times \mathbb{R} | t \le 1 - 2\epsilon(x)\}$ and $\lambda_2^{-1}(0) = \{(x,t) \in X \times \mathbb{R} | t \ge 1 - \epsilon(x)\}$.

Let Ω be a representation of G containing Y as a definable C^rG submanifold. By Lemma 4.12, there exist a G invariant definable open neighborhood U of Y in Ω and a definable C^rG map $p : U \to Y$ such that $p|Y = id_Y$.

If H' is a sufficiently close approximation of H and ϵ is a sufficiently small positive G invariant definable function, then $\tilde{H} : X \times \mathbb{R} \to Y$,

$$\tilde{H}(x,t) = \begin{cases} p((1-\lambda_1(t))f_1(x) + \lambda_1(t)H'(x,t)), & (x,t) \in X \times (-\infty, \frac{1}{2}] \\ p((1-\lambda_2(t))f_2(x) + \lambda_2(t)H'(x,t)), & (x,t) \in X \times [\frac{1}{2}, \infty) \end{cases}$$

is a definable C^rG map such that $\tilde{H}(x,0) = f_1(x)$ and $\tilde{H}(x,1) = f_2(x)$ for all $x \in X$. Therefore f_1 is definably C^rG homotopic to f_2.

Now we prove (2). Let η be a G vector bundle over X of rank k. Then by Theorem 1.3, η is G vector bundle isomorphic to a strongly definable G vector bundle η' over X. Thus we can find a representation Ξ of G and a definable G map $f : X \to G(\Xi, k)$ such that η' is definably G vector bundle isomorphic to $f^*(\gamma(\Xi, k))$. By Theorem 4.11, we have a definable C^rG map $F : X \to G(\Xi, k)$ as an approximation of f. If this approximation is sufficiently close, then f is definably G homotopic to F. Hence by Proposition 4.5 and Theorem 1.3, $f^*(\gamma(\Xi, k))$ is definably G vector bundle isomorphic to $F^*(\gamma(\Xi, k))$. Therefore η is G vector bundle isomorphic to a (strongly) definable C^rG vector bundle $F^*(\gamma(\Xi, k))$.

Let ζ_1 and ζ_2 be definable C^rG vector bundles over X which are G vector bundle isomorphic. Then by Theorem 1.3, they are definably G vector bundle isomorphic. Thus by Proposition 4.14, they are definably C^rG vector bundle isomorphic. Therefore (2) is proved. □

References

[1] M. F. Atiyah, *K-theory*, Benjamin, 1967.
[2] E. Bierstone, *The equivariant covering homotopy property for differentiable G-fibre bundles*, J. Diff. Geom. **8** (1973), 615–622.
[3] J. Bochnak, M. Coste and M.F. Roy, *Géométie algébrique réelle*, Springer-Verlag (1987).
[4] H. Delf and M. Knebusch, *Separation, retraction and homotopy extension in semialgebraic spaces*, Pacific J. Math. **114** (1) (1984), 47–71.
[5] L. van den Dries, *Tame topology and o-minimal structures*, Lecture notes series **248**, London Math. Soc. Cambridge Univ. Press (1998).
[6] L. van den Dries, A. Macintyre, and D. Marker, *Logarithmic-exponential power series*, J. London. Math. Soc., II. Ser. **56**, No.3 (1997), 417-434.
[7] L. van den Dries, A. Macintyre, and D. Marker, *The elementary theory of restricted analytic field with exponentiation*, Ann. Math. **140** (1994), 183–205.
[8] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497-540.
[9] L. van den Dries and P. Speissegger, *The real field with convergent generalized power series*, Trans. Amer. Math. Soc. **350**, (1998), 4377–4421.
[10] T. Kawakami, *Algebraic G vector bundles and Nash G vector bundles*, Chinese J. Math. **22** (1994), 275–289.
[11] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology Appl. **123** (2002), 323-349.
[12] T. Kawakami, *Imbedding of manifolds defined on an o-minimal structures on* $(\mathbb{R}, +, \cdot, <)$, Bull. Korean Math. Soc. **36** (1999), 183–201.
[13] R. K. Lashof, *Equivariant Bundles*, Illinois J. Math. **26(2)** (1982), 257–271.
[14] D.H. Park and D.Y. Suh, *Equivariant semialgebraic homotopies*, Topology Appl. **115** (2001), 153-174.
[15] D.H. Park and D.Y. Suh, *Semialgebraic G CW complex structure of semialgebraic G spaces*, J. Korean Math. Soc. **35** (1998), 371–386.
[16] Y. Peterzil, A. Pillay and S. Starchenko, *Definably simple groups in o-minimal structures*, Trans. Amer. Math. Soc. **352** (2000), 4397–4419.
[17] A. Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Appl. Algebra **53** (1988), 239-255.
[18] G. Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 129–151.
[19] A. Tarski, *A decision method for elementary algebra and geometry*, 2nd edition. revised, Berkeley and Los Angeles (1951).

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