EQUIVARIANT DEFINABLE C^r APPROXIMATION THEOREM, DEFINABLE C^rG TRIVIALITY OF G INVARIANT DEFINABLE C^r FUNCTIONS AND COMPACTIFICATIONS

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Received August 2, 2004

ABSTRACT. Let G be a compact subgroup of $GL_n(\mathbb{R})$ and $0 \leq s < r < \infty$. We prove that every definable C^sG map between affine definable C^rG manifolds is approximated in the definable C^s topology by definable C^rG maps. We show that each G invariant proper submersive surjective definable C^r function defined on an affine definable C^rG manifold is definably C^rG trivial. Moreover we prove that every noncompact affine definable C^rG manifold admits a unique affine definable C^rG compactification up to definable C^rG diffeomorphism when $r \geq 2$.

1. INTRODUCTION

By [14] if s is a non-negative integer, then every C^s Nash map between affine Nash manifolds is approximated in the C^s topology by Nash maps. This C^s topology is a new topology defined in [14] which is different from the C^s Whitney topology in general. There is a generalization of this result in the definable C^r category obtained by an o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ on the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers, namely if $0 \leq s < r < \infty$, then every definable C^s map between affine definable C^r manifolds is approximated in the definable C^s topology by definable C^r maps (II.5.2 [15]). This definable C^s topology is useful to approximate definable C^s maps between affine definable C^r manifolds by definable C^r maps, and it is a generalization of the C^s topology defined in [14]. Approximations of maps between affine definable C^s manifolds are with respect to the definable C^s topology, unless otherwise stated. The Nash category coincides with the definable category based on $\mathcal{R}$ [17], and definable categories based on $\mathcal{M}$ are generalizations of the Nash category. General references on o-minimal structures are [3], [5], see also [15]. Further properties and constructions of them are studied in [4], [6], [13].

In an arbitrary o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ of $\mathcal{R}$, we are concerned with an equivariant version of the above result in [15], definable C^rG triviality of G invariant proper submersive surjective definable C^r functions and definable C^rG compactifications of noncompact affine definable C^rG manifolds.

The term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{M}$" and every definable map is assumed to be continuous. In this paper, G denotes a compact subgroup of $GL_n(\mathbb{R})$ and any manifold does not have boundary, unless otherwise

2000 *Mathematics Subject Classification*. 57S15, 14P20, 57R35, 58A07, 03C64.
Keywords and Phrases. Definable C^rG manifolds, definable C^rG maps, approximation theorem, proper, definably C^rG trivial, definable C^rG vector bundles, compactifications.

stated. Under our assumption, G is a compact algebraic subgroup of $GL_n(\mathbb{R})$ (e.g. 2.2 [12]). We now list the main results of this paper.

Theorem 1.1. *If $0 \leq s < r < \infty$, then every definable C^sG map between affine definable C^rG manifolds is approximated in the definable C^s topology by definable C^rG maps.*

The following is existence of a definable C^rG tubular neighborhood of a definable C^rG submanifold of a representation of G when $1 \leq r < \infty$.

Proposition 1.2. *If $1 \leq r < \infty$, then every definable C^rG submanifold X of a representation Ω of G has a definable C^rG tubular neighborhood (U, θ) of X in Ω, namely U is a G invariant definable open neighborhood of X in Ω and $\theta : U \to X$ is a definable C^rG map with $\theta|X = id_X$.*

Note that if $r = \infty$ or ω, then Proposition 1.2 is already known in [9].

Suppose that η is a definable C^rG vector bundle over an affine definable C^rG manifold X and $1 \leq r \leq \omega$. We say that η is *strongly definable* if there exist a representation Ω of G and a definable C^rG map $f : X \to G(\Omega, \alpha)$ such that η is definably C^rG vector bundle isomorphic to $f^*(\gamma(\Omega, \alpha))$, where α denotes the rank of η.

Proposition 1.3. *Let X, Y be affine definable C^rG manifolds and $1 \leq r < \infty$.*

(1) *X and Y are definably C^1G diffeomorphic if and only if they are definably C^rG diffeomorphic.*

(2) *Let η_1 and η_2 be strongly definable C^rG vector bundles over X. Then they are definably G vector bundle isomorphic if and only if they are definably C^rG vector bundle isomorphic.*

Note that if G is a finite group and $0 \leq r < \infty$, then every definable C^rG vector bundle is strongly definable (1.8 [8]). Moreover a definable $C^\infty G$ vector bundle over an affine definable $C^\infty G$ manifold is strongly definable if and only if its total space is affine (4.14 [8]).

Let X be a definable C^rG manifold, $f : X \to \mathbb{R}$ a G invariant surjective definable C^r function and $1 \leq r \leq \omega$. We say that f is *definably C^rG trivial* if there exist a definable C^rG manifold F and a definable C^rG map $h : X \to F$ such that the map $H : X \to \mathbb{R} \times F$ defined by $H = (f, h)$ is a definable C^rG diffeomorphism. If f is definably C^rG trivial, then for every $y \in Y$, the fiber $f^{-1}(y)$ of y is a definable C^rG submanifold of X which is definably C^rG diffeomorphic to F. Hence one can find a definable C^rG diffeomorphism $\phi : X \to \mathbb{R} \times f^{-1}(y)$ such that $f = p \circ \phi$, where p denotes the projection $\mathbb{R} \times f^{-1}(y) \to \mathbb{R}$.

A map $\psi : M \to N$ between topological spaces is called *proper* if for any compact subset C of N, $\psi^{-1}(C)$ is compact.

The following is an equivariant definable C^r version of [1].

Theorem 1.4. *Let X be an affine definable C^rG manifold and $1 \leq r < \infty$. Then every G invariant proper submersive surjective definable C^r function $f : X \to \mathbb{R}$ is definable C^rG trivial.*

The following is a result on existence and uniqueness of affine definable C^rG compactifications of a noncompact affine definable C^rG manifold when $2 \leq r < \infty$.

Theorem 1.5. *Let X be a noncompact affine definable C^rG manifold and $1 \leq r < \infty$.*

(1) [1.2 [8]] *There exists a compact affine definable C^rG manifold Y with boundary such that the interior of Y is definably C^rG diffeomorphic to X.*
(2) *If Z is another compact affine definable C^rG manifold with boundary whose interior is definably C^rG diffeomorphic to X and $r \geq 2$, then Z is definably C^rG diffeomorphic to Y.*

This paper is organized as follows. In Section 2, we recall definable C^rG manifolds and definable C^rG vector bundles ([9], [8]) and list several results required in the proof of our results. We prove Theorem 1.1 - Proposition 1.3 in Section 3 and Theorem 1.4 and 1.5 in Section 4.

2. Definable C^rG manifolds and definable C^rG vector bundles

Recall the definition of definable C^rG manifolds ([9], [8]).

Definition 2.1 ([9], [8]). Let $0 \leq r \leq \omega$.

(1) A group homomorphism (resp. A group isomorphism) from G to $O_n(\mathbb{R})$ is a *definable group homomorphism* (resp. a *definable group isomorphism*) if it is a definable map (resp. a definable homeomorphism).
Note that a definable group homomorphism (resp. a definable group isomorphism) between G and $O_n(\mathbb{R})$ is a definable C^∞ map (resp. a definable C^∞ diffeomorphism) because G and $O_n(\mathbb{R})$ are Lie groups.
(2) An *n-dimensional representation* of G means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from G to $O_n(\mathbb{R})$. In this paper, we assume that every representation of G is orthogonal.
(3) A *definable C^rG manifold* is a pair (X, α) consisting of a definable C^r manifold X and a group action α of G on X such that $\alpha : G \times X \to X$ is a definable C^r map. For simplicity of notation, we write X instead of (X, α).
(4) A definable C^r submanifold of a definable C^rG manifold X is called a *definable C^rG submanifold* of X if it is G invariant.
(5) A definable C^r map (resp. A definable C^r diffeomorphism, A definable homeomorphism, A definable map) is a *definable C^rG map* (resp. a *definable C^rG diffeomorphism*, a *definable G homeomorphism*, a *definable G map*) if it is a G map.
(6) A definable C^rG manifold is called *affine* if it is definably C^rG diffeomorphic (definably G homeomorphic if $r = 0$) to a definable C^rG submanifold of some representation of G.
(7) A *definable C^rG manifold with boundary* is defined similarly.

If $\mathcal{M}$ is polynomially bounded and $0 \leq r < \infty$, then every definable C^r manifold is affine [9], and if $\mathcal{M}$ is exponential, then each compact definable $C^\infty G$ manifold is affine [9].

Recall the definition of definable C^rG vector bundles [8].

Definition 2.2 ([8]). Suppose that $0 \leq r \leq \omega$.

(1) A *definable C^rG vector bundle* is a definable C^r vector bundle $\eta = (E, p, X)$ satisfying the following three conditions.
(a) The total space E and the base space X are definable C^rG manifolds.

(b) The projection $p : E \to X$ is a definable C^rG map.
(c) For any $x \in X$ and $g \in G$, the map $p^{-1}(x) \to p^{-1}(gx)$ is linear.

(2) Let η and ζ be definable C^rG vector bundles over X. A definable C^r vector bundle morphism $\eta \to \zeta$ is called a *definable C^rG vector bundle morphism* if it is a G map. A definable C^rG vector bundle morphism $f : \eta \to \zeta$ is said to be a *definable C^rG vector bundle isomorphism* if there exists a definable C^rG vector bundle morphism $h : \zeta \to \eta$ such that $f \circ h = id$ and $h \circ f = id$.

(3) A definable C^r section of a definable C^rG vector bundle is a *definable C^rG section* if it is a G map.

(4) If $r = 0$, then a definable C^0G vector bundle (resp. a definable C^0G vector bundle morphism, a definable C^0G vector bundle isomorphism, a definable C^0G section) is simply called a *definable G vector bundle* (resp. a *definable G vector bundle morphism*, a *definable G vector bundle isomorphism*, a *definable G section*).

Recall the definable C^s topology [8] and three results on it [8].

Let X and Y be definable C^s submanifolds of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $0 \leq s < \infty$. Let $C^s_{def}(X, Y)$ denote the set of definable C^s maps from X to Y. For $f \in C^s_{def}(X, Y)$ and $x \in X$, the differential df_x of f at x means a linear map from the tangent space T_xX of X at x to $\mathbb{R}^m$. Composing it with the orthogonal projection $\mathbb{R}^n \to T_xX$, one can extend df_x to a linear map $\mathbb{R}^n \to \mathbb{R}^m$. Then $Df : X \to M(m, n; \mathbb{R}) = \mathbb{R}^{mn}$ is defined as the matrix representation of df. For each $1 \leq k \leq s$, we inductively define a C^{s-k} map

$$D^k f : X \to \mathbb{R}^{n^k m}, D^k f = D(D^{k-1} f).$$

Let $||f||_s$ denote the definable function on X defined by

$$||f||_s(x) = |f(x)| + |Df(x)| + \cdots + |D^s f(x)|.$$

For a positive definable function $\epsilon : X \to \mathbb{R}$, let

$$U_\epsilon = \{h \in C^s_{def}(X, Y) | ||h||_s < \epsilon\}.$$

We say that the *definable C^s topology* on $C^s_{def}(X, Y)$ is the topology defined by choosing $\{h + U_\epsilon\}_\epsilon$ as a fundamental neighborhood system of h in $C^s_{def}(X, Y)$. In the Nash category, we simply call it the *C^r topology*. If X is compact, then this topology coincides with the C^s Whitney topology (p 156 [15]).

Proposition 2.3 ([15], 4.9 [8]). *Let X, Y and Z be definable C^s submanifolds $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^l$, respectively, and $0 \leq s < \infty$. Let $f \in C^s_{def}(X, Y)$ and $h \in C^s_{def}(Y, Z)$.*

(1) *The map $h_* : C^s_{def}(X, Y) \to C^s_{def}(X, Z), h_*(k) = h \circ k$ is continuous.*

(2) *The map $f^* : C^s_{def}(Y, Z) \to C^s_{def}(X, Z), f^*(k) = k \circ f$ is continuous if and only if f is proper.*

Proposition 2.4 ([15], 4.10 [8]). *Let X and Y be definable C^s submanifolds of $\mathbb{R}^n$ and $0 < s < \infty$. Let $f : X \to Y$ be a definable C^s map. If f is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image), then an approximation of f in the definable C^s topology is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image). Moreover if f is a diffeomorphism, then $h^{-1} \to f^{-1}$ as $h \to f$.*

Theorem 2.5 ([15], 4.11 [8]). *Let X and Y be affine definable C^r manifolds and $0 \le s < r < \infty$. Then every definable C^s map $f : X \to Y$ is approximated in the definable C^s topology by definable C^r maps.*

By the proof of 2.10 [8], we have the following.

Proposition 2.6 (2.10 [8]). *Let X be a definable C^rG submanifold of a representation Ω of G and $0 \le r < \infty$. Then X admits a closed definable C^rG imbedding into $\Omega \times \mathbb{R}$.*

The proof of 4.8 [8] proves the following.

Proposition 2.7 (4.8 [8]). *(Definable C^r partition of unity). Let X be a definable closed subset of $\mathbb{R}^n$, $\{U_i\}_{i=1}^l$ a finite definable open covering of X and $0 \le r < \infty$. Then there exist definable C^r functions $\lambda_1, \dots, \lambda_l : \mathbb{R}^n \to \mathbb{R}$ such that $0 \le \lambda_i \le 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

Recall universal G vector bundles (e.g. [8]) and existence of a Nash G tubular neighborhood of a Nash G submanifold of a representation of G (2.3 [10]).

Definition 2.8. Let Ω be an n-dimensional representation of G induced by a definable group homomorphism $B : G \to O_n(\mathbb{R})$. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \to B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer α, we define the vector bundle $\gamma(\Omega, \alpha) = (E(\Omega, \alpha), u, G(\Omega, \alpha))$ as follows:

$$G(\Omega, \alpha) = \{A \in M(\Omega) | A^2 = A, A = A', TrA = \alpha\},$$

$$E(\Omega, \alpha) = \{(A, v) \in G(\Omega, \alpha) \times \Omega | Av = v\},$$

$$u : E(\Omega, \alpha) \to G(\Omega, \alpha), u((A, v)) = A,$$

where A' denotes the transposed matrix of A and $Tr\ A$ stands for the trace of A. Then $\gamma(\Omega, \alpha)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, \alpha)$ is algebraic, it is an algebraic G vector bundle. We call it *the universal G vector bundle associated with Ω and α*. Remark that $G(\Omega, \alpha) \subset M(\Omega)$ and $E(\Omega, \alpha) \subset M(\Omega) \times \Omega$ are nonsingular algebraic G sets. In particular, they are Nash G submanifolds of $M(\Omega)$ and $M(\Omega) \times \Omega$, respectively.

Proposition 2.9 (2.3 [10]). *Every Nash G submanifold X of a representation Ω of G has a Nash G tubular neighborhood (U, θ) of X in Ω.*

The following is the definable C^r cell decomposition theorem (e.g. 7.3.3 [3]).

Theorem 2.10 (e.g. 7.3.3 [3]). *(Definable C^r cell decomposition). Let $1 \le r < \infty$.*

(1) *For any definable sets $A_1, \dots, A_k \subset \mathbb{R}^n$, there exists a decomposition of $\mathbb{R}^n$ into definable C^r cells partitioning $A_1, \dots, A_k$.*

(2) *For every definable function $f : A \to \mathbb{R}. A \subset \mathbb{R}^n$, there exists a decomposition of $\mathbb{R}^n$ into definable C^r cells partitioning A such that each restriction $f|C : C \to \mathbb{R}$ is of class C^r for each definable C^r cell $C \subset A$ of the decomposition.*

Theorem 2.10 remains valid in a more general setting (e.g. 7.3.3. [3]).

Let G be a compact Lie group, f a map from a C^rG manifold X to a representation Ω of G and $0 \le r \le \infty$. Denote the Haar measure of G by dg, and let x be a point in X. Recall the *averaging operator* A defined by

$$A(f)(x) = \int_G g^{-1} f(gx) dg.$$

Proposition 2.11 (4.1 [2]). *Let G be a compact Lie group and $0 \le r \le \infty$. Suppose that $C^r(X, \Omega)$ denotes the set of C^r maps from a C^rG submanifold X of a representation of G to a representation Ω of G.*

(1) *The averaged map $A(f)$ of f is equivariant, and $A(f) = f$ if f is equivariant.*
(2) *If $f \in C^r(X, \Omega)$, then $A(f) \in C^r(X, \Omega)$.*
(3) *If f is a polynomial map, then so is $A(f)$.*
(4) *If X is compact and $r < \infty$, then $A : C^r(X, \Omega) \to C^r(X, \Omega)$ is continuous in the C^r Whitney topology.*

3. Proof of Theorem 1.1 - Proposition 1.3

Recall existence of definable C^∞ slices [9].

Theorem 3.1 ([9]). *Let G be a compact affine definable C^∞ group, X a definable $C^\infty G$ manifold and $x \in X$. Then there exists a linear definable C^∞ slice at x in X.*

To prove Theorem 1.1, we need a definable C^r version of Theorem 3.1.

Proposition 3.2. *Let X be a definable C^rG submanifold of a representation Ω of G and $1 \le r < \infty$. Then for any $x \in X$, there exists a linear definable C^r slice at x in X, namely there exists a definable C^rG_x imbedding i from a representation Ξ of G_x into X such that $i(0) = x$, $G \times_{G_x} \Xi$ is a definable C^rG manifold with the standard action $(g, [g', x]) \mapsto [gg', x]$ and the map $\mu : G \times_{G_x} \Xi \to X$ defined by $[g, x] \mapsto gi(x)$ is a definable C^rG diffeomorphism onto some G invariant definable open neighborhood of $G(x)$ in X.*

Proof. Since G is a compact algebraic subgroup of $GL_n(\mathbb{R})$ and by Theorem 3.1, for any $x \in X$, there exists a linear definable C^∞ slice at x in Ω, namely we have a representation Ξ' of G_x and a definable $C^\infty G_x$ imbedding $j : \Xi' \to \Omega$ such that $j(0) = x$, $G \times_{G_x} \Xi'$ is a definable $C^\infty G$ manifold and the map $\mu' : G \times_{G_x} \Xi' \to \Omega$ defined by $\mu'([g, x]) = gj(x)$ is a definable $C^\infty G$ diffeomorphism onto a G invariant definable open neighborhood $Gj(\Xi')$ of $G(x)$ in Ω. Then $j^{-1}(X)$ is a definable C^rG_x submanifold of Ξ' and $j|j^{-1}(X) : j^{-1}(X) \to X$ is a definable C^rG_x imbedding. Hence there exists a sufficiently small G_x invariant definable open neighborhood U of 0 in $j^{-1}(X)$ such that U is definably C^rG_x diffeomorphic to a representation Ξ of G_x. Take a definable C^rG_x diffeomorphism $l : \Xi \to U$ with $l(0) = 0$ and let $i = j \circ l$. Then i is a definable C^rG_x imbedding from Ξ to X and the map $\mu : G \times_{G_x} \Xi \to X$ defined by $\mu([g, x]) = gi(x)$ is a definable C^rG diffeomorphism onto a G invariant definable open neighborhood $Gi(\Xi) = Gj(U)$ of $G(x)$ in X. $\square$

In a way similar to usual $C^\infty G$ manifold cases (e.g. 4.19 [11]), we have the following proposition.

Proposition 3.3. *Let X be an affine definable C^rG manifold, H closed subgroup of G and $1 \leq r < \infty$. Then the union $M_X(H)$ of the orbits of type (G/H) is a definable C^rG submanifold of X.*

A similar proof of 1.4 [8] proves the following.

Proposition 3.4. *Let X be an affine definable C^rG manifold with only one orbit type and $1 \leq r < \infty$. Then the orbit space X/G admits an affine definable C^r manifold structure such that :*

(1) The orbit map $\pi : X \to X/G$ is a G invariant submersive surjective definable C^r map.

(2) For any map f from X/G to any affine definable C^r manifold Y, f is a definable C^r map if and only if so is $f \circ \pi$.

The following is a result on piecewise definable C^rG triviality of G invariant submersive surjective definable C^r maps [8].

Theorem 3.5 (1.1 [8]). *(Piecewise definable C^rG triviality). Let X be an affine definable C^rG manifold, Y an affine definable C^r manifold and $1 \leq r < \infty$. Suppose that $f : X \to Y$ is a G invariant submersive surjective definable C^r map. Then there exist a finite decomposition $\{T_i\}_{i=1}^k$ of Y into definable C^r submanifolds and definable C^rG diffeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(y_i)$ such that $f|f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \leq i \leq k)$, where p_i denotes the projection $T_i \times f^{-1}(y_i) \to T_i$ and $y_i \in T_i$.*

The following is an equivariant version of Proposition 2.7.

Proposition 3.6. *(Equivariant definable C^r partition of unity). Let X be a definable C^rG submanifold closed in a representation Ω of G and $\{U_i\}_{i=1}^l$ a finite G invariant definable open covering of X and $0 \leq r < \infty$. Then there exist G invariant definable C^r functions $\lambda_1, \dots, \lambda_l : X \to \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

Proof. First of all, we recall the structure of the orbit space Ω/G. The algebra $\mathbb{R}[\Omega]^G$ of G invariant polynomials on Ω is finitely generated [18]. Let $p_1, \dots, p_n : \Omega \to \mathbb{R}$ be G invariant polynomials generating $\mathbb{R}[\Omega]^G$, and put $p : \Omega \to \mathbb{R}^n, p = (p_1, \dots, p_n)$. Then p is a proper polynomial map, and it induces a closed imbedding $j : \Omega/G \to \mathbb{R}^n$ such that $p = j \circ \pi$, where $\pi : \Omega \to \Omega/G$ denotes the orbit map. Hence we can identify Ω/G (resp. X/G, π) with $j(\Omega/G)$ (resp. $j(X/G)$, p). Thus $\{p(U_i)\}_{i=1}^l$ is a finite definable open covering of X/G because $p|X : X \to X/G$ is open. Note that $p(X)$ is closed in $\mathbb{R}^n$ because X is closed in Ω. By Proposition 2.7, one can find a definable partition of unity $\{\overline{\lambda}_i\}_{i=1}^l$ subordinate to $\{p(U_i)\}_{i=1}^l$. Hence $\lambda_1 := \overline{\lambda}_1 \circ p, \dots, \lambda_l := \overline{\lambda}_l \circ p$ are the required G invariant definable C^r functions. □

The following is a weaker version of Theorem 1.1.

Proposition 3.7. *If $0 \leq s < r < \infty$, then every definable C^sG map from an affine definable C^rG manifold X to a representation Ξ of G is approximated in the definable C^s topology by definable C^rG maps.*

We now prove Proposition 3.7 and 1.2 simultaneously.

Proof of Proposition 3.7 *and* 1.2. Let Ω be a representation of G containing X as a definable C^rG submanifold. By Proposition 2.6, we may assume that X is closed in Ω.

We prove two results by induction on $\dim X$ and the number of connected components of X. If $\dim X = 0$, then X consists of finitely many points. Thus Proposition 3.7 is clearly true. Since X is a definable $C^\infty G$ manifold and by [8], Proposition 1.2 holds.

As in the proof of Proposition 3.6, there exists a proper polynomial map $p : \Omega \to \mathbb{R}^n$ (resp. $q : \Xi \to \mathbb{R}^m$) such that $p|X : X \to X/G \subset \mathbb{R}^n$ (resp. $q : \Xi \to \Xi/G \subset \mathbb{R}^m$) is the orbit map of X (resp. Ξ).

Let $f : X \to \Xi$ be a definable C^sG map. Then f induces a definable map $\overline{f} : X/G \to \Xi/G$ with $q \circ f = \overline{f} \circ (p|X)$. Note that Ξ has only finitely many orbit types. Let $(G/H_1), \dots, (G/H_s)$ be the orbit types of Ξ. For each (G/H_i), by Proposition 3.3, $M_\Xi(H_i)$ is a definable C^rG submanifold of Ξ. Using Proposition 3.4, the orbit space $M_\Xi(H_i)/G$ of $M_\Xi(H_i)$ is a definable C^r submanifold of $\mathbb{R}^m$ and the orbit map $q|M_\Xi(H_i) : M_\Xi(H_i) \to M_\Xi(H_i)/G$ is a G invariant surjective submersive definable C^r map. Similarly, we have a finite partition of X/G into definable C^r submanifolds $\{M_X(K_j)/G\}_{j=1}^t$ such that each $p|M_X(K_j) : M_X(K_j) \to M_X(K_j)/G$ is a G invariant surjective submersive definable C^r map.

By Theorem 3.5, for each i, there exist a finite partition of $M_\Xi(H_i)/G$ into definable C^r submanifolds $\{W_{ik}\}_{k=1}^{u_i}$ of $M_\Xi(H_i)/G$ and definable C^rG diffeomorphisms $\psi_{ik} : q^{-1}(W_{ik}) \to W_{ik} \times q^{-1}(b_{ik})$, $(1 \le k \le u_i)$ such that $q|q^{-1}(W_{ik}) = proj_{ik} \circ \psi_{ik}$, $(1 \le k \le u_i)$, where $b_{ik} \in W_{ik}$ and $proj_{ik}$ denotes the projection $W_{ik} \times q^{-1}(b_{ik}) \to W_{ik}$.

Since each $\overline{f}^{-1}(W_{ik})$ is a definable subset of X/G and by Theorem 2.10, there exists a finite decomposition $\{C_l\}_{l=1}^v$ of X/G into definable C^r cells partitioning $\{M_X(K_j)/G\}_{j=1}^t$ and $\{\overline{f}^{-1}(W_{ik})\}_{1 \le i \le s, 1 \le k \le u_i}$. Then by construction of $\{C_l\}_{l=1}^v$, each $p^{-1}(C_l)$ is a definable C^rG submanifold of some $M_X(K_j)$ and $p|p^{-1}(C_l) : p^{-1}(C_l) \to C_l$ is a G invariant surjective submersive definable C^r map. Hence applying Theorem 3.5 to each $p|p^{-1}(C_l) : p^{-1}(C_l) \to C_l$, we have a finite partition $\{Z_{l\alpha}\}_{\alpha=1}^{w_l}$ of C_l into definable C^r submanifolds of C_l such that for each $Z_{l\alpha}$ there exist a definable C^rG diffeomorphism $\phi_{l\alpha} : p^{-1}(Z_{l\alpha}) \to Z_{l\alpha} \times p^{-1}(a_{l\alpha})$ with $p|p^{-1}(Z_{l\alpha}) = proj'_{l\alpha} \circ \phi_{l\alpha}$, where $a_{l\alpha} \in Z_{l\alpha}$ and $proj'_{l\alpha}$ denotes the projection $Z_{l\alpha} \times p^{-1}(a_{l\alpha}) \to Z_{l\alpha}$. Hence

$$(f^1_{ikl\alpha}, f^2_{ikl\alpha}) := \psi_{ik} \circ f \circ \phi_{l\alpha}^{-1} : Z_{l\alpha} \times p^{-1}(a_{l\alpha}) \to W_{ik} \times q^{-1}(b_{ik})$$

is a definable C^sG map such that $f^1_{ikl\alpha} : Z_{l\alpha} \to W_{ik}$ is a definable C^s map. If $\dim p^{-1}(a_{l\alpha}) < \dim X$, then the inductive hypothesis produces a definable C^rG map $h^2_{ikl\alpha} : p^{-1}(a_{l\alpha}) \to q^{-1}(b_{ik})$ approximating $f^2_{ikl\alpha} : p^{-1}(a_{l\alpha}) \to q^{-1}(b_{ik})$. If $\dim p^{-1}(a_{l\alpha}) = \dim X$, then $p^{-1}(a_{l\alpha})$ is a union of connected components of X because $p^{-1}(a_{l\alpha})$ is open and closed in X. If $p^{-1}(a_{l\alpha}) = X$, then G acts on X transitively. By Theorem 2.10, $f^2_{ikl\alpha} : p^{-1}(a_{l\alpha}) \to q^{-1}(b_{ik})$ is of class C^r at some point in $p^{-1}(a_{l\alpha})$. Since the action is transitive, $f^2_{ikl\alpha} : p^{-1}(a_{l\alpha}) \to q^{-1}(b_{ik})$ is a definable C^rG map. If $p^{-1}(a_{l\alpha}) \ne X$, then by the inductive hypothesis, we have a definable C^rG map $h^2_{ikl\alpha} : p^{-1}(a_{l\alpha}) \to q^{-1}(b_{ik})$ approximating $f^2_{ikl\alpha} : p^{-1}(a_{l\alpha}) \to q^{-1}(b_{ik})$. By Theorem 2.5, we have a definable C^r map $h^1_{ikl\alpha} : Z_{l\alpha} \to W_{ik}$ as an approximation of $f^1_{ikl\alpha} : Z_{l\alpha} \to W_{ik}$. Thus

$$(h^1_{ikl\alpha}, h^2_{ikl\alpha}) : Z_{l\alpha} \times p^{-1}(a_{l\alpha}) \to W_{ik} \times q^{-1}(b_{ik})$$

is a definable C^rG map approximating $(f^1_{ikl\alpha}, f^2_{ikl\alpha}) : Z_{l\alpha} \times p^{-1}(a_{l\alpha}) \to W_{ik} \times q^{-1}(b_{ik})$. Hence by Proposition 2.3, there exists a definable C^rG map $h_{l\alpha} : p^{-1}(Z_{l\alpha}) \to \Xi$ as an approximation of $f|p^{-1}(Z_{l\alpha}) : p^{-1}(Z_{l\alpha}) \to \Xi$. If $p^{-1}(Z_{l\alpha})$ is not open in X, then $\dim p^{-1}(Z_{l\alpha}) < \dim X$. Using the inductive hypothesis of Proposition 1.2, there exist a definable C^rG tubular neighborhood $(Z'_{l\alpha}, p_{l\alpha})$ of $p^{-1}(Z_{l\alpha})$ in Ω. Thus there exist a G invariant definable open neighborhood $Z''_{l\alpha}$ of $p^{-1}(Z_{l\alpha})$ in X and a definable C^rG map $h'_{l\alpha} : Z''_{l\alpha} \to \Xi$ approximating $f|Z''_{l\alpha} : Z''_{l\alpha} \to \Xi$. By Proposition 3.6 and since X is closed in Ω, we can glue these maps. Therefore we have the required definable C^rG map $h : X \to \Xi$.

We now prove Proposition 1.2. Let $F : X \to G(\Omega, \beta)$ be the classifying map of the normal bundle of X in Ω, where β denote the codimension of X in Ω. Then F is a definable $C^{r-1}G$ map. Applying Proposition 3.7 to $I \circ F : X \to M(\Omega)$, we have a definable C^rG map $\overline{H} : X \to M(\Omega)$ as an approximation of $I \circ F$, where I denotes the inclusion $G(\Omega, \beta) \to M(\Omega)$. By Proposition 2.9, there exists a Nash G tubular neighborhood of $G(\Omega, \beta)$ in $M(\Omega)$. If our approximation is sufficiently close, composing the projection of this Nash G tubular neighborhood, we have a definable C^rG map $H : X \to G(\Omega, \beta)$ approximating $F : X \to G(\Omega, \beta)$. Moreover $H(x) + T_xX = T_x(\Omega)$ for all $x \in X$ because $F(x) + T_xX = T_x(\Omega)$ for all $x \in X$. Thus $L := \{(x, y) \in X \times \Omega | y \in H(x)\}$ is a definable C^rG submanifold of $X \times \Omega$. Let $\overline{\theta} : L \to \Omega, \overline{\theta}(x, y) = x + y$. Then there exists a G invariant positive definable function $\epsilon : \Omega \to \mathbb{R}$ such that the restriction of $\overline{\theta}$ to $L_\epsilon := \{(x, y) \in L | ||y|| < \epsilon(x)\}$ is a definable C^rG imbedding and $U := \overline{\theta}(L_\epsilon)$ is a G invariant definable open neighborhood of X in Ω, where $||y||$ denotes the standard norm of y in Ω. Therefore U and $\theta := \Phi \circ (\overline{\theta}|L_\epsilon)^{-1}$ fulfill the requirements, where $\Phi : L_\epsilon \to X, \Phi(x, y) = x$. □

Proof of Theorem 1.1. Let $f : X \to Y$ be a definable C^sG map and Ξ a representation of G containing Y as a definable C^rG submanifold. Then by Proposition 3.7, there exists a definable C^rG map $H : X \to \Xi$ as an approximation of $I \circ f$, where I denotes the inclusion $Y \to \Xi$. By Proposition 1.2, we have a definable C^rG tubular neighborhood (U, θ) of Y in Ξ. If our approximation is sufficiently close, then the image of H lies in U. Therefore $\theta \circ H : X \to Y$ is the required definable C^rG map. □

To prove Proposition 1.3, we need the following.

Proposition 3.8 (4.5 [8]). *Let η and ζ be strongly definable C^rG vector bundles over an affine definable C^rG manifold and $0 \le r \le \omega$. Then $Hom(\eta, \zeta)$ is also a strongly definable definable C^rG vector bundle.*

Proof of Proposition 1.3. (1) Let f be a definable C^1G diffeomorphism between affine definable C^rG manifolds X and Y. By Theorem 1.1, there exists a definable C^rG map $h : X \to Y$ as an approximation of f. If this approximation is sufficiently close, then by Proposition 2.4 and the inverse function theorem, h is the required definable C^rG diffeomorphism.

(2) Since η_1 and η_2 are strongly definable and by Proposition 3.8, $\mathrm{Hom}(\eta_1, \eta_2)$ is a strongly definable C^rG vector bundle over X. Thus there exist a representation Ω_1 of G and a definable C^rG map $f_1 : X \to G(\Omega_1, \alpha_1)$ such that $\mathrm{Hom}(\eta_1, \eta_2)$ is definably C^rG vector bundle isomorphic to $f_1^*(\gamma(\Omega_1, \alpha_1))$, where α_1 denotes the rank of $\mathrm{Hom}(\eta_1, \eta_2)$. By assumption, there exists a definable G vector bundle isomorphism between η_1 and η_2. Hence it defines a definable G section of $\mathrm{Hom}(\eta_1, \eta_2)$ which lies in $\mathrm{Iso}(\eta_1, \eta_2)$. Thus

this section induces a definable G map $s' : X \to \Omega_1$ such that $f_1(x)s'(x) = s'(x)$ for any $x \in X$. By Theorem 1.1, there exists a definable C^rG map $s'' : X \to \Omega_1$ as an approximation of s'. Thus $s(x) := f_1(x)s''(x)$ is a definable C^rG section of $\mathrm{Hom}(\eta_1, \eta_2)$ because $f_1(x)s(x) = f_1^2(x)s''(x) = f_1(x)s''(x) = s(x)$ for any $x \in X$. If this approximation is sufficiently close, then s lies in $\mathrm{Iso}(\eta_1, \eta_2)$. Therefore it defines a definable C^rG vector bundle isomorphism between η_1 and η_2. □

4. Proof of Theorem 1.4 and 1.5

The following example shows that the proper condition cannot be removed in Theorem 1.4 even if $G = 1$.

Example 4.1. *If $X = \{(x,y) \in \mathbb{R}^2 | y = -1\} \cup \{(x,y) \in \mathbb{R}^2 | xy = 1, x > 0\} \subset \mathbb{R}^2$ and $f : X \to \mathbb{R}, f(x,y) = x$, then f is a submersive surjective definable C^ω map, and it is piecewise definably C^ω trivial but not definably trivial.*

Proof of Theorem 1.4. Applying Theorem 3.5, we have a partition $-\infty = a_0 < a_1 < a_2 < \cdots < a_j < a_{j+1} = \infty$ of $\mathbb{R}$ and definable C^rG diffeomorphisms $\phi_i : f^{-1}((a_i, a_{i+1})) \to (a_i, a_{i+1}) \times f^{-1}(y_i)$ with $f|f^{-1}((a_i, a_{i+1})) = p_i \circ \phi_i$, $(0 \le i \le j)$, where p_i denotes the projection $(a_i, a_{i+1}) \times f^{-1}(y_i) \to (a_i, a_{i+1})$ and $y_i \in (a_i, a_{i+1})$.

Now we prove that for each a_i with $1 \le i \le j$, there exist an open interval I_i containing a_i and a definable C^rG map $\pi_i : f^{-1}(I_i) \to f^{-1}(a_i)$ such that $F_i = (f, \pi_i) : f^{-1}(I_i) \to I_i \times f^{-1}(a_i)$ is a definable C^rG diffeomorphism. By Proposition 1.2, we have a definable C^rG tubular neighborhood (U_i, π_i) of $f^{-1}(a_i)$ in X. Since f is proper, there exists an open interval I_i containing a_i such that $f^{-1}(I_i) \subset U_i$. Note that Example 4.1 shows that if f is not proper, then such an open interval does not always exist. Hence shrinking I_i, if necessary, $F_i = (f, \pi_i) : f^{-1}(I_i) \to I_i \times f^{-1}(a_i)$ is the required definable C^rG diffeomorphism.

By the above argument, we have a finite family of $\{J_i\}_{i=1}^l$ of open intervals and definable C^rG diffeomorphisms $h_i : f^{-1}(J_i) \to J_i \times f^{-1}(y_i)$, $(1 \le i \le l)$, such that $y_i \in J_i$, $\cup_{i=1}^l J_i = \mathbb{R}$ and the composition of h_i with the projection $J_i \times f^{-1}(y_i)$ onto J_i is $f|f^{-1}(J_i)$.

Now we glue these trivializations to get a global one. We can suppose that $i \ge 2$, $U_{i-1} \cap J_i = (a,b)$ and $k_{i-1} : f^{-1}(U_{i-1}) \to U_{i-1} \times f^{-1}(y_1)$ is a definable C^rG diffeomorphism with $f|f^{-1}(U_{i-1}) = proj_{i-1} \circ k_{i-1}$, where $U_{i-1} = \cup_{s=1}^{i-1} J_s$ and $proj_{i-1}$ denotes the projection $U_{i-1} \times f^{-1}(y_1) \to U_{i-1}$. Take $z \in (a,b) = U_{i-1} \cap J_i$. Then since $f^{-1}(y_1) \cong f^{-1}(z) \cong f^{-1}(y_i)$, $f^{-1}(y_1)$ is definably C^rG diffeomorphic to $f^{-1}(y_i)$. Hence we may assume that h_i is a definable C^rG diffeomorphism from $f^{-1}(J_i)$ to $J_i \times f^{-1}(y_1)$. Then we have a definable C^rG diffeomorphism

$$k_{i-1} \circ h_i^{-1} : (a,b) \times f^{-1}(y_1) \to (a,b) \times f^{-1}(y_1), (t,x) \mapsto (t, q(t,x)).$$

Take a C^r Nash function $u : \mathbb{R} \to \mathbb{R}$ such that $u = \frac{a+b}{2}$ on $(-\infty, \frac{3}{4}a + \frac{1}{4}b]$ and $u = id$ on $[\frac{1}{4}a + \frac{3}{4}b, \infty)$. Let

$$H : (a,b) \times f^{-1}(y_1) \to f^{-1}((a,b)), H(t,x) = k_{i-1}^{-1}(t, q(u(t), x)).$$

Then H is a definable C^rG diffeomorphism such that $H = h_i^{-1}$ if $\frac{1}{4}a + \frac{3}{4}b \le t \le b$ and $H = k_{i-1}^{-1} \circ (id \times \psi)$ if $a \le t \le \frac{3}{4}a + \frac{1}{4}b$, where $\psi : f^{-1}(y_1) \to f^{-1}(y_1), \psi(x) = q(\frac{a+b}{2}, x)$.

Thus we can define

$$\tilde{k}_i : f^{-1}(U_i) \to U_i \times f^{-1}(y_1),$$

$$\tilde{k}_i(x) = \begin{cases} (id \times \psi)^{-1} \circ k_{i-1}(x), & f(x) \le \frac{3}{4}a + \frac{1}{4}b \\ H^{-1}(x), & \frac{3}{4}a + \frac{1}{4}b \le f(x) \le b \\ h_i(x), & f(x) > b \end{cases}.$$

Then $\tilde{k}_i$ is a definable C^rG diffeomorphism. Hence if our approximation is sufficiently close, then $k_i = (f, P) : f^{-1}(U_i) \to U_i \times f^{-1}(y_1)$ is a definable C^rG diffeomorphism. Therefore k_l is the required definable C^rG diffeomorphism. □

The following is an equivariant definable version of VI.2.2 [16], which proves Theorem 1.5 (2).

Theorem 4.2. *Let X and Y be compact affine definable C^rG manifolds possibly with boundary and $2 \le r < \infty$. Then the following three conditions are equivalent.*

(1) *X and Y are C^1G diffeomorphic.*
(2) *X and Y are definably C^rG diffeomorphic.*
(3) *The interior of X is definably C^rG diffeomorphic to that of Y.*

The next two results are equivariant definable versions of 4.1 (3) [1] and VI.I.4 [16].

Proposition 4.3. *Let X be a noncompact affine definable C^1G manifold. If $f, h : X \to \mathbb{R}$ are G invariant proper positive definable C^1 functions, then there exists a C^1G diffeomorphism $\tau : X \to X$ such that $h \circ \tau = f$ outside a G invariant compact definable subset of X.*

Proof. Assume that X is a definable C^1G submanifold of some representation of G.

At first we prove that there exists some $a > 0$ such that $f|f^{-1}((a,\infty)) : f^{-1}((a,\infty)) \to (a,\infty)$ is submersive.

Let $Z := \{x \in X | x \text{ is a critical point of } f\}$. Then Z is a definable subset of X. Applying Theorem 2.10, Z admits a finite partition $U_1, \dots, U_l$ into definable C^1 cells. Take $x, y \in U_1$. Since U_1 is a definable C^1 cell, there exists a definable C^1 curve $\gamma : [u,v] \to U_1$ such that $\gamma(u) = x$ and $\gamma(b) = y$. Then $f \circ \gamma : [u,v] \to \mathbb{R}$ is a definable C^1 function whose derivative is identically zero because each point in U_1 is a critical point of f. Hence $f \circ \gamma$ is a constant function, in particular $f(x) = f(y)$. Thus f is constant on U_1. Therefore there exists the required positive number a because $f(Z)$ consists of at most l points.

We now prove that there exists a G invariant compact definable subset K of X such that $\lambda df(x) + (1-\lambda)dh(x) \ne 0$ for all $x \in X - K$ and for all $\lambda \in [0,1]$.

If such a compact subset would not exist, there would be a definable curve in X, going to infinity, and on which f and h have derivatives whose product is negative or null. This would contradict the assumption that f and h are proper, positive and G invariant.

Consider the function H on X defined by $H(x) = f(x) + (c+1+h(x)-f(x)) \cdot \psi(f(x)-c)$, where ψ is a C^1 Nash function on $\mathbb{R}$ such that ψ is equal to 0 on a neighborhood of $(-\infty, 0]$ and to 1 on a neighborhood of $[1,\infty)$, and that the derivative $\psi' \ge 0$. Then H coincides with $h + c + 1$ outside a G invariant compact definable set $f^{-1}([0, c+1])$. The constant c is chosen such that $f^{-1}([0,c)) \supset K$ and $c > a$. Put $\mu(x) = \psi(f(x) - c)$. Then $dH(x) = (1-\mu(x))df(x) + \mu(x)dh(x) + (c+1+h(x)-f(x))\psi'(f(x)-c)df(x)$, and it is never null outside K because the coefficients of $df(x)$ and $dh(x)$ are always positive or null and never simultaneously null. Thus H is proper and $H|Y : Y \to \mathbb{R}$ is submersive, where $Y = H^{-1}([c,\infty)] = f^{-1}([c,\infty))$. By Theorem 1.4 and since $f^{-1}(c) = H^{-1}(c)$, we have definable C^1G diffeomorphisms $\sigma_1, \rho : f^{-1}(c) \times [c,\infty) \to Y$ such that $H \circ \sigma_1$ and $f \circ \rho$ are the projection $f^{-1}(c) \times [c,\infty) \to [c,\infty)$.

Take a C^1 Nash diffeomorphism $s : [c,\infty) \to [c,\infty)$ such that $s(x) = x$ for all $x \in [c, c+\frac{1}{3}]$ and $s(x) = x + c + 1$ for all $x \in [c+\frac{2}{3},\infty)$. Then $\sigma = \sigma_1 \circ (id_{f^{-1}(c)} \times s)$ is a C^1G diffeomorphism such that $h \circ \sigma$ coincides with the projection $f^{-1}(c) \times [c,\infty) \to [c,\infty)$ outside a G invariant compact definable subset of $f^{-1}(c) \times [c,\infty)$.

We can extend $\sigma \circ \rho^{-1}$ to a C^1G diffeomorphism $\tau : X \to X$ by setting $\tau = id$ on $f^{-1}([0,c))$, and τ has the required property. □

Proposition 4.4. *Let X be a compact affine definable C^1G manifold with boundary ∂X. Suppose that h_1, h_2 are G invariant non-negative definable functions on X such that $h_1(0) = h_2(0) = \partial X$ and $h_1|Int\ X$ and $h_2|Int\ X$ are G invariant definable C^1 functions. Then there exists a positive number ϵ such that $\{x \in X | h_1(x) \geq \epsilon\}$ is C^1G diffeomorphic to $\{x \in X | h_2(x) \geq \epsilon\}$.*

Proof. Let $f_i : \mathrm{Int}\ X \to \mathbb{R}, f_i := \frac{1}{h_i}$, $i = 1, 2$. Then f_1 and f_2 are G invariant proper positive definable C^1 functions. By Proposition 4.3, there exist a G invariant compact subset K of Int X and a C^1G diffeomorphism $\tau : \mathrm{Int}\ X \to \mathrm{Int}\ X$ such that $f_2 = f_1 \circ \tau$ on Int $X - K$. Thus there exists a positive number k such that $\tau|\{x \in X | f_1(x) \geq k\} : \{x \in X | f_1(x) \geq k\} \to \{x \in X | f_2(x) \geq k\}$ is a C^1G diffeomorphism. Taking $\epsilon := \frac{1}{k}$, we have a C^1G diffeomorphism $\tau|\{x \in X | 0 < h_1(x) \leq \epsilon\} : \{x \in X | 0 < h_1(x) \leq \epsilon\} \to \{x \in X | 0 < h_2(x) \leq \epsilon\}$. Therefore $\tau|\{x \in X | h_1(x) \geq \epsilon\} : \{x \in X | h_1(x) \geq \epsilon\} \to \{x \in X | h_2(x) \geq \epsilon\}$ is the required C^1G diffeomorphism. □

The following is an equivariant definable C^r version of I.3.2 [16].

Proposition 4.5. *Let X be a compact definable C^rG submanifold possibly with boundary of a representation Ω of G and $1 \leq r < \infty$. Then there exists a definable C^rG tubular neighborhood (U, θ) of X in Ω.*

Proof. As in the proof of Proposition 1.2, the classifying map $f : \mathrm{Int}\ X \to G(\Omega, \alpha)$ of the normal bundle of Int X in Ω is a definable $C^{r-1}G$ map, where α denotes the codimension of Int X in Ω. Since the graph of the classifying map $F : X \to G(\Omega, \alpha)$ of the normal bundle of X in Ω is the closure of that of f in $X \times G(\Omega, \alpha)$, F is definable. Thus F is a definable $C^{r-1}G$ map. Since X is compact and by the polynomial approximation theorem, Proposition 2.11 and 2.9, we have a definable C^rG map $h : X \to G(\Omega, \alpha)$ as an approximation of F. Therefore a similar proof of Proposition 1.2 proves the result. □

Proposition 4.6. *Let X be a compact affine definable C^rG manifold with boundary and $2 \leq r < \infty$. Then X admits a definable C^rG collar, namely there exists a definable C^rG imbedding $\phi : \partial X \times [0,1] \to X$ such that $\phi|(\partial X \times \{0\})$ is the inclusion $\partial X \to X$, where the action on $[0,1]$ is trivial.*

Proof. Let Ω be a representation of G containing X as a definable C^rG submanifold of Ω. By Proposition 1.2, there exists a definable C^rG tubular neighborhood of ∂X in Ω. Using this definable C^rG tubular neighborhood and the averaging process, a similar proof of 4.6.1 [7] proves that X admits a C^rG collar, namely there exists a C^rG imbedding $\rho_1 : \partial X \times [0,1] \to X$ such that $\rho_1|(\partial X \times \{0\})$ is the inclusion $\partial X \to X$.

Let $\rho_2 : \partial X \times [0,1] \to X, \rho_2(x,t) = x$. Then $\rho_1 - \rho_2 = 0$ on $\partial X \times \{0\}$. Hence $\rho_1(x,t) - \rho_2(x,t) = \int_0^1 (\frac{\partial}{\partial u}(\rho_1(x,tu) - \rho_2(x,tu)))du = t\int_0^1 (\frac{\partial \rho_1}{\partial t}(x,tu) - \frac{\partial \rho_2}{\partial t}(x,tu))du$. Thus there exists a $C^{r-1}G$ map $\rho_3 : \partial X \times [0,1] \to \Omega$ such that $\rho_1(x,t) - \rho_2(x,t) = t\rho_3(x,t)$.

By Proposition 4.5, there exists a definable C^rG tubular neighborhood (U,θ) of X in Ω. Since $r \geq 2$ and by the polynomial approximation theorem and Proposition 2.11, we can find a polynomial G map $\rho_4 : \partial X \times [0,1] \to \Omega$ as an approximation of ρ_3 in the C^1 Whitney topology. Then $\phi = \theta(\rho_2 + t\rho_4)$ is a definable C^rG map approximating ρ_1 in the C^1 Whitney topology. If our approximation is sufficiently close, then ϕ is the required definable C^rG imbedding. □

Proof of Theorem 4.2. Let Ω (resp. Ξ) be a representation of G containing X (resp. Y) as a definable C^rG submanifold of Ω (resp. Ξ).

By the polynomial approximation theorem, Proposition 2.11 and 1.2, for two compact affine definable C^rG manifold without boundary, they are C^1G diffeomorphic if and only if they are definably C^rG diffeomorphic. Thus assume that $\partial X \neq \emptyset$ and $\partial Y \neq \emptyset$.

(3) $\Rightarrow$ (1). Let $F : \text{Int } X \to \text{Int } Y$ be a definable C^rG diffeomorphism. By Proposition 4.6, there exists a definable C^rG collar $\phi_X : \partial X \times [0,1] \to X$ (resp. a definable C^rG collar $\phi_Y : \partial Y \times [0,1] \to Y$) of ∂X in X (resp. of ∂Y in Y). Using these collars, we have non-negative G invariant definable C^r functions $h_1 : X \to \mathbb{R}$ and $h_2 : Y \to \mathbb{R}$ such that $h_1^{-1}(0) = \partial X$, $h_2^{-1}(0) = \partial Y$ and h_1, h_2 are C^1 regular at $\partial X, \partial Y$, respectively. Thus there exists a sufficiently small number $\epsilon > 0$ such that X (resp. Y) is C^1G diffeomorphic to $X_\epsilon = \{x \in X | h_1(x) \geq \epsilon\}$ (resp. $Y_\epsilon = \{x \in Y | h_2(x) \geq \epsilon\}$).

A G invariant definable C^r function $h_2 \circ F$ is extendable to X as a G invariant definable function whose zero set is ∂X. By Proposition 4.4, replacing $\epsilon > 0$, if necessary, X_ϵ and $\{x \in X | h_2 \circ F(x) \geq \epsilon\}$ are C^1G diffeomorphic. Since $F(\{x \in X | h_2 \circ F(x) \geq \epsilon\}) = Y_\epsilon$, X is C^1G diffeomorphic to Y.

(1) $\Rightarrow$ (2). Let $f : X \to Y$ be a C^1G diffeomorphism. Since $f|\partial X : \partial X \to \partial Y$ is a C^1G diffeomorphism and ∂X is compact, as in the second paragraph, one can find a definable C^rG diffeomorphism $f' : \partial X \to \partial Y$ as an approximation of $f|\partial X : \partial X \to \partial Y$ in the C^1 Whitney topology. Using definable C^rG collars of ∂X and ∂Y in X and Y, respectively, we have a G invariant definable open neighborhoods U and V of ∂X and ∂Y in X and Y, respectively, and a definable C^rG diffeomorphism $f_1 : U \to V$ with $f_1|\partial X = f'$.

Take a G invariant definable open neighborhood U' of ∂X in X with $U' \subsetneq U$. Then there exists a G invariant definable C^r function $\lambda : X \to \mathbb{R}$ such that $\lambda = 1$ on U' and its support lies in U. By Proposition 4.5 and since Y is compact, there exists a definable C^rG tubular neighborhood (V,θ) of Y in Ξ. By the polynomial approximation theorem, Proposition 2.11 and since X is compact, there exists a polynomial G map $f_2 : X \to \Omega$ which is an approximation of $I \circ f$ in the C^1 Whitney topology, where $I : Y \to \Xi$ denotes the inclusion. If our approximation is sufficiently close, then

$$H : X \to Y, H(x) = \theta(\lambda(x)f_1(x) + (1 - \lambda(x))f_2(x))$$

is a definable C^rG map such that it is an approximation of f in the C^1 Whitney topology and $H(\partial X) \subset \partial Y$.

Recall that the fact that the set of C^1 diffeomorphisms from X to Y is open with respect to the C^1 Whitney topology in $\{\psi|\psi : X \to Y \text{ is a } C^1 \text{ map with } \psi(\partial X) \subset \partial Y\}$ (e.g. p38 [7]). Therefore by the inverse function theorem, H is the required definable C^rG diffeomorphism.

The implication (2) $\Rightarrow$ (3) is trivial. □

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