LOCALLY DEFINABLE C^sG MANIFOLD STRUCTURES OF LOCALLY DEFINABLE C^rG MANIFOLDS

TOMOHIRO KAWAKAMI

Received July 26, 2005

ABSTRACT. Let G be a finite group. We define locally definable C^rG $(1 \leq r \leq \omega)$ manifolds as generalizations of definable C^rG manifolds $(1 \leq r \leq \omega)$. Let $0 < r < s < \infty$. We prove that every affine locally definable C^rG manifold is locally definably C^rG diffeomorphic to a locally definable C^sG manifold. Moreover we prove that for any two affine locally definable C^rG manifolds, they are C^1G diffeomorphic if and only if they are locally definable C^rG diffeomorphic.

1. INTRODUCTION

Let G be a finite group and $0 < r < s < \infty$. In this paper we are concerned with existence of locally definable C^sG manifold structures of an affine locally definable C^rG manifold and uniqueness of affine locally definable C^sG manifold structures up to locally definable C^sG diffeomorphism. All locally definable C^rG manifolds are considered in the o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, >, \dots)$ of $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$.

A *locally definable* C^r *manifold* is a C^r manifold admitting a countable system of charts whose gluing maps are of class definable C^r. If this system is finite, then it is called a *definable* C^r *manifold*. Definable C^rG manifolds are studied in [4], [5], [6], [7], [8]. A locally definable C^r manifold is *affine* if it can be imbedded into some $\mathbb{R}^n$ in a locally definable C^r way. We can define locally definable C^rG manifolds and affine locally definable C^rG manifolds in a similar way of equivariant definable cases. Locally definable C^rG manifolds are generalizations of definable C^rG manifolds which are studied in [6].

In the present paper G denotes a finite group and every manifold does not have boundary unless otherwise stated.

Theorem 1.1. *Let G be a finite group and $0 < r < s < \infty$. Then every affine locally definable C^rG manifold is locally definably C^rG diffeomorphic to some locally definable C^sG manifold.*

Theorem 1.2. *Let G be a finite group and let r be a positive integer. Then for any two affine locally definable C^rG manifolds, they are C^1G diffeomorphic if and only if they are locally definably C^rG diffeomorphic.*

Even in the non-equivariant Nash category, if $r = \infty$, then Theorem 1.2 does not hold because there exist two affine Nash manifolds such that they are not Nash diffeomorphic but C^∞ diffeomorphic [11], and that for any two affine Nash manifolds, they are locally Nash diffeomorphic if and only if they are Nash diffeomorphic.

Existence of $C^\omega G$ manifold structures of proper $C^\infty G$ manifolds and uniqueness of them are studied by S.Illman in [2] and [3], respectively, when G is a C^ω Lie group.

1991 *Mathematics Subject Classification*: 14P10, 14P15, 14P20, 57S05, 57S15, 58A07.

Key Words and Phrases: Locally definable C^rG manifolds, definable C^rG manifolds, o-minimal, finite groups .

Moreover F. Kutzschebauch [10] showed that if G is a compact C^ω Lie group, then for any two $C^\omega G$ manifolds, they are $C^\infty G$ diffeomorphic if and only if they are $C^\omega G$ diffeomorphic. Theorem 1.1 and 1.2 are locally definable C^r versions of [2] and [3], respectively, when G is a finite group.

It is known that if G is a compact affine Nash group, then a compact $C^\infty G$ manifold is $C^\infty G$ diffeomorphic to some affine Nash G manifold, and that it admits a uncountable family of structures of nonaffine Nash G manifolds if it is connected and of positive dimension, and the action on it is not transitive [9]. Moreover a $C^\infty G$ manifold admits an affine Nash G manifold structure if and only if it is either compact or compactifiable [9]. Here a $C^\infty G$ manifold is *compactifiable* if it is $C^\infty G$ diffeomorphic to the interior of some compact $C^\infty G$ manifold with boundary. Our theorems are locally definable C^r versions of results of [9].

In the non-equivariant setting, we have the following.

Theorem 1.3. *If r is a positive integer, then every n-dimensional locally definable C^r manifold X is locally definably C^r imbeddable into $\mathbb{R}^{2n+1}$.*

The above theorem is the locally definable version of Whitney's imbedding theorem (e.g. 2.14 [1]). The definable version of Theorem 1.1 is known in [7].

By Theorem 1.1 and 1.3, we have the following theorem.

Theorem 1.4. *Let $1 < r < s < \infty$. Then every locally definable C^r manifold admits a unique affine locally definable C^s manifold structure up to locally definable C^s diffeomorphism.*

This paper is organized as follows. In section 2 we define locally definable C^rG manifolds and we state preliminary results for the proof of our theorems. We prove our results in section 3, 4 and 5.

2. Locally definable C^rG manifolds

A subset X of $\mathbb{R}^n$ is called *locally definable* if for every $x \in X$ there exists a definable open neighborhood U of x in $\mathbb{R}^n$ such that $X \cap U$ is definable in $\mathbb{R}^n$. Clearly every definable set is locally definable. Remark that every compact locally definable set is definable, and that any open subset of $\mathbb{R}^n$ is locally definable.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be locally definable sets. We call a map $f : U \to V$ *locally definable* if the graph of f ($\subset U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$) is locally definable.

For example, if $\mathcal{M} = \mathbf{R}_{an}$, then $f : (-1,1) \to \mathbb{R}, f(x) = \sin \frac{1}{1-x^2}$ is locally definable but not definable.

Note that for any continuous locally definable map f between locally definable sets X and Y, if X is compact, then $f(X)$ is a definable set and $f : X \to f(X)$ ($\subset Y$) is a definable map.

Notice that for every locally definable map f between locally definable sets X and Y and for any $x \in X$, there exist a definable open neighborhood $U' \subset X$ of x and a definable open neighborhood $V' \subset Y$ of $f(x)$ such that $f(U') \subset V'$ and $f|U' : U' \to V'$ is definable. Moreover the converse holds true.

Remark that the maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \sin x, f_2(x) = \cos x$, respectively, are analytic but not locally definable in $\mathbf{R} = (\mathbb{R}, +, \cdot, >)$. Remark further that the field $\mathbb{Q}$ ($\subset \mathbb{R}$) of rational numbers is not a locally definable subset of $\mathbb{R}$.

Proposition 2.1. *(1) Let X, Y and Z be locally definable sets and let $f : X \to Y$ and $g : Y \to Z$ be locally definable maps. If f is continuous, then $g \circ f : X \to Z$ is locally definable.*
(2) Let $\{X_j\}_{j=1}^\infty$ be a locally finite family of locally definable subsets of a locally definable set L. Then $\cup_{j=1}^\infty X_i$ is locally definable. If h is a map from L to a locally definable set L' such that each $h|X_j$ is locally definable, and that $h|(\cup_{j=1}^\infty X_j) : \cup_{j=1}^\infty X_j \to L'$ is injective, then $h|(\cup_{j=1}^\infty X_j)$ is locally definable.

Proof. (1) Since $f : X \to Y$ is locally definable, for any $x \in X$, there exist a definable open neighborhood U of x in X and a definable open neighborhood V of $f(x)$ in Y such that $f(U) \subset V$ and $f|U : U \to V$ is definable. By the same reason, there exist a definable open neighborhood V' of $f(x)$ in Y and a definable open neighborhood W of $g \circ f(x)$ in Z such that $g(V') \subset W$ and $g|V' : V' \to W$ is definable. Since $V \cap V'$ is a definable open neighborhood of $f(x)$ in Y and since f is continuous, $U' := (f|U)^{-1}(V \cap V')$ is a definable open neighborhood of x in X. Thus $g \circ f|U' : U' \to W$ is the composition of definable maps $f|U'$ and $g|V \cap V'$. Hence $g \circ f|U'$ is a definable map. Therefore $g \circ f$ is locally definable.

Now we show (2). Let $x \in \cup_{j=1}^{\infty} X_j$. Since $\cup_{j=1}^{\infty} X_j$ is a locally finite union, there exists an open neighborhood $\hat{U}$ of x such that $\hat{U}$ intersects only finitely many X_j. We may assume that $\hat{U} \cap X_j \neq \emptyset$ $(1 \le j \le k)$ and $\hat{U} \cap X_j = \emptyset$ $(j > k)$. Since each X_j is locally definable, there exist definable open neighborhoods U_j $(1 \le j \le k)$ of x such that $U_j \cap X_j$ is definable and $U_j \subset \hat{U}$. Thus $(\cap_{j=1}^{k} U_j) \cap (\cup_{j=1}^{\infty} X_j) = \cup_{l=1}^{\infty}((\cap_{j=1}^{k} U_j) \cap X_l) = \cup_{l=1}^{k}((\cap_{j=1}^{k} U_j) \cap X_l)$ is definable. Therefore $\cup_{j=1}^{\infty} X_j$ is locally definable. The latter half of (2) follows from the first one because the graph of $h|(\cup_{j=1}^{\infty} X_j)$ is a locally finite union of that of $h|X_j$. □

Remark 2.2. (1) Every locally definable set is a countable locally finite union of definable sets because a (locally definable) subset of $\mathbb{R}^n$ is paracompact. Thus the closure of a locally definable set is locally definable by Proposition 2.1 (2), and the interior of it is locally definable because it is open.
(2) The union of an infinite family of definable sets is not always locally definable. A subset $M = \cup_{n=1}^{\infty}\{(x,y) \in \mathbb{R}^2 | y = nx\}$ of $\mathbb{R}^2$ is not locally definable because for any open definable neighborhood U of the origin in $\mathbb{R}^2$, $M \cap U$ is not definable. This shows that in the first half of Proposition 2.1 (2) one cannot drop the locally finite condition on $\{X_j\}$.
(3) The projection image of a locally definable set is not always locally definable. Let $N = \cup_{n=1}^{\infty}\{(x,y,z) \in \mathbb{R}^3 | y = nx, z = n\}$ and let p denote the projection $\mathbb{R}^3 \to \mathbb{R}^2, p(x,y,z) = (x,y)$. Then N is a locally definable set, $M = p(N)$ is not locally definable, and $p|N : N \to \mathbb{R}^2$ is not locally definable. This example also shows that in the latter half of Proposition 2.1 (2) we cannot drop the injective condition on $h|(\cup_{j=1}^{\infty} Z_j)$.
(4) The complement of a locally definable set is not necessarily locally definable. A subset of $\mathbb{R}^2 - \overline{M}$ $(= \mathbb{R}^2 - (M \cup \{(x,y) \in \mathbb{R}^2 | x = 0\}))$ of $\mathbb{R}^2$ is locally definable because it is open. However $\overline{M}$ is not locally definable.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. A C^r map $f : U \to V$ is called a *locally definable C^r map* if f is locally definable. A locally definable C^r map $f : U \to V$ is called a *locally definable C^r diffeomorphism* if there exists a locally definable C^r map $h : V \to U$ such that $f \circ h = id$ and $h \circ f = id$.

We now define locally definable C^r manifolds.

Definition 2.3. Let $1 \le r \le \omega$.
(1) A locally definable subset X of $\mathbb{R}^n$ is called a *d-dimensional locally definable C^r submanifold of* $\mathbb{R}^n$ if for any $x \in X$ there exists a definable C^r diffeomorphism ϕ from some definable open neighborhood U of the origin in $\mathbb{R}^n$ onto some definable open neighborhood V of x in $\mathbb{R}^n$ such that $\phi(0) = x, \phi(\mathbb{R}^d \cap U) = X \cap V$. Here $\mathbb{R}^d = \{x \in \mathbb{R}^n |$ last $(n-d)$ components of x are zero.$\}$
(2) A *locally definable C^r manifold X of dimension d* is a C^r manifold with a countable system of charts $\{\phi_i : U_i \to \mathbb{R}^d\}$ such that for each i and j $\phi_i(U_i \cap U_j)$ is a definable open subset of $\mathbb{R}^d$ and the map $\phi_j \circ \phi_i^{-1} | \phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. We call these atlas *locally definable C^r*. Locally definable C^r manifolds with compatible atlases are identified. Clearly every definable C^r manifold is a locally definable C^r manifold. A subset Y of a locally definable C^r manifold X is called a *k-dimensional locally definable C^r submanifold of X* if each point $x \in Y$ there exists a locally definable C^r chart $\phi_i : U_i \to \mathbb{R}^d$ of X such that $x \in U_i$ and $U_i \cap Y = \phi_i^{-1}(\mathbb{R}^k)$, where $\mathbb{R}^k \subset \mathbb{R}^d$ is the vectors whose last $(d-k)$ components are zero. Remark that every point x of a locally definable C^r manifold

W admits a definable open neighborhood U of x in W such that U is a definable C^r manifold. We call U a *definable C^r neighborhood* of x.
(3) Let h be a C^r map between locally definable C^r manifolds M and N. We say that h is a *locally definable C^r map* if for any $x \in M$, there exist definable C^r neighborhoods of U_1 of x in M and U_2 of $f(x)$ in N such that $f(U_1) \subset U_2$ and $f|U_1 : U_1 \to U_2$ is a definable C^r map.
(4) Let X and Y be locally definable C^r manifolds. We say that X and Y are *locally definably C^r diffeomorphic* if one can find locally definable C^r maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h = id$ and $h \circ f = id$.
(5) A locally definable C^r manifold is said to be *affine* if it is locally definably C^r diffeomorphic to a locally definable C^r submanifold of some Euclidean space $\mathbb{R}^l$.

Example 2.4. (1) An open subset of $\mathbb{R}^n$ and a countable disjoint union of definable manifolds are locally definable manifolds.
(2) Remark that a locally definable C^∞ manifold is not always a locally definable C^ω manifold. For example, if $\mathcal{M} = \mathbf{R}_{an}$, then the graph of the function defined by $f : \mathbb{R} \to \mathbb{R}, \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$ is a locally definable C^∞ manifold but not of class definable C^ω.

We can define locally definable groups and affine locally definable groups in a similar way of definable cases. But we do not give their definitions here because we restrict our attention to finite groups.

A *representation map* of G is a group homomorphism from G to some $GL(\mathbb{R}^n)$. This map can be seen as the restriction of a polynomial map $\mathbb{R} \to GL(\mathbb{R}^n)$ $(\subset \mathbb{R}^{n^2})$ because G is a finite group. A *representation* of G means the representation space of a representation map of G.

Definition 2.5. Let $1 \leq r \leq \omega$.
(1) A locally definable C^r submanifold of a representation Ω of G is called a *locally definable C^rG submanifold of Ω* if it is G invariant.
(2) A *locally definable C^rG manifold* is a pair (X, θ) consisting of a locally definable C^r manifold X and a group action θ of G on X such that $\theta : G \times X \to X$ is a locally definable C^r map. For simplicity of notation, we write X instead of (X, θ). Clearly each definable C^rG manifold is a locally definable C^rG manifold.
(3) Let X and Y be locally definable C^rG manifolds. A locally definable C^r map is called a *locally definable C^rG map* if it is a G map. We say that X and Y are *locally definably C^rG diffeomorphic* if there exist locally definable C^rG maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h = id$ and $h \circ f = id$.
(4) A locally definable C^rG manifold is said to be *affine* if it is locally definably C^rG diffeomorphic to a locally definable C^rG submanifold of some representation of G.

Remark that we can define locally definable G manifolds for a locally definable group G, but in the present paper we do not use these notions.

Recall existence of definable C^rG tubular neighborhoods.

Proposition 2.6. *([5]) Let X be an affine definable C^rG submanifold of a representation Ω of G and $0 < r < \infty$. Then there exists a definable C^rG tubular neighborhood (U, p) of X in Ω, namely U is an affine definable C^rG submanifold of Ω and the projection $p : U \to X$ is a definable C^rG map.*

Let $G = \{g_1, \ldots, g_n\}$ and let f be a C^rG map from a C^rG manifold M to a representation Ω of G. Then the averaging map $A(f) : M \to \Omega$ is

$$A(f)(x) = \frac{1}{n} \sum_{i=1}^{n} g_i^{-1} f(g_i x).$$

Proposition 2.7. *([6]) (1) $A(f)$ is equivariant, and $A(f) = f$ if f is equivariant.*
(2) If f is a polynomial map, then so is $A(f)$.
(3) If $0 \leq r \leq \infty$ and f lies in the set $C^r(M, \Omega)$ of C^r maps from M to Ω, then

$A(f) \in C^r(M, \Omega)$.
(4) $A : C^r(M, \Omega) \to C^r(M, \Omega), f \mapsto A(f)$ $(0 \leq r < \infty)$ *is continuous in the* C^r *Whitney topology.*
(5) If M *is a definable* C^rG *manifold,* f *is a definable* C^r *map, and* $1 \leq r \leq \omega$*, then* $A(f)$ *is a definable* C^rG *map.*

Let K be a subgroup of G. Suppose that S is an affine definable C^rK manifold. Then we know that the twisted product $G \times_K S$ with the standard action $G \times (G \times_K S) \to G \times_K S, (g, [g', s]) \mapsto [gg', s]$ is a definable C^rG manifold [5].

We need the following proposition to prove Theorem 1.1.

Proposition 2.8. *Let* X *be a locally definable* C^rG *manifold. Suppose that* K *is a subgroup of* G *and* N *is an affine definable* C^rK *manifold. If* $f : N \to X$ *is a locally definable* C^rK *map, then*

$$\mu(f) : G \times_K N \to X, \mu([g, n]) = gf(n)$$

is a locally definable C^rG *map.*

Proof. By the property of quotient manifolds, $\mu(f)$ is a C^rG map. Thus it suffices to prove that $\mu(f)$ is locally definable. Let π be the orbit map $G \times N \to G \times_K N$. Then π is a definable C^r map. Take $x \in G \times_K N$ and $y \in \pi^{-1}(x) \subset G \times N$. By the assumption and the definition of the G action on $G \times N$, $\overline{\mu}(f) : G \times N \to X, \overline{\mu}(f)(g, n) = gf(n)$ is a locally definable C^rG map. Hence there exist definable open C^r neighborhood U of y and V of $\overline{\mu}(f)(y)$ such that $\overline{\mu}(f)(U) \subset V$ and $\overline{\mu}(f)|U : U \to V$ is a definable C^r map. In particular, $\overline{\mu}(f)|U : U \to V$ is definable. Hence $\pi(U)$ is open and definable. Since the graph of $\mu(f)|\pi(U) : \pi(U) \to V \subset X$ is the image of that of $\overline{\mu}(f)|U$ by $\pi \times id_V$, $\mu(f)|\pi(U)$ is definable. □

We can consider definable C^r slices as well as smooth ones.

Definition 2.9. Let X be a definable C^rG manifold and $0 < r < \infty$.
(1) We say that a K invariant definable C^r submanifold S of X is a *definable K slice* if GS is open in X, S is affine as a definable C^rK manifold, and

$$\mu : G \times_K S \to GS \ (\subset X), [g, x] \mapsto gx$$

is a definable C^rG diffeomorphism.
Remark that μ is always definable because its graph is the image of that of $G \times S \to GS, (g, s) \mapsto gs$ by $\pi \times id_{GS}$, where π denotes the orbit map $G \times S \to G \times_K S$.
(2) A definable C^rK slice S is called *linear* if there exist a representation Ω of K and a definable C^rK imbedding $j : \Omega \to X$ such that $j(\Omega) = S$.
(3) We say that a definable C^rK slice (resp. a linear definable C^rK slice) S is a *definable* C^r *slice* (resp. a *linear definable* C^r *slice*) *at* x *in* X if $K = G_x$ and $x \in S$ (resp. $K = G_x$, $x \in S$ and $j(0) = x$).

Recall existence of definable C^r slices [5] to prove Theorem 1.2 .

Theorem 2.10. *([5]) Let* X *be an affine definable* C^rG *manifold,* $x \in X$ *and* $0 < r < \infty$*. Then there exists a linear definable* C^rG *slice at* x *in* X*.*

3. Proof of Theorem 1.1

The following lemma is obtained by 2.2.8 [1] and Proposition 2.7.

Lemma 3.1. *Let* K *be a finite group. Suppose that* f *is a definable* C^rK *map between definable* C^rK *manifolds* M *and* N*. Suppose further that* V *is an open* K *invariant subset of* M *and that* P *is a* K *invariant definable* C^r *submanifold of* N *with* $f(V) \subset P$*. Then there exist an open neighborhood* $\mathfrak{N}$ *of* $f|V$ *in the set* $Def^r_K(V, P)$ *of definable* C^rK *maps from* V *to* P *such that for any* $h \in \mathfrak{N}$*, the map*

$$E(h) : M \to N,$$

$$E(h)(x) = \begin{cases} h(x), & x \in V \\ f(x), & x \in M - V \end{cases}$$

is a definable C^rK *map and* $E : \mathfrak{N} \to Def^r_K(M, N), h \mapsto E(h)$ *is continuous in the* C^r *Whitney topology.*

Proposition 3.2. *Let* X *be a locally definable* C^rG *manifold,* Y *an affine definable*

C^rG manifold in a representation Ω of G and $0 \le r < \infty$. Then every C^rG map $f : X \to Y$ is approximated by a locally definable C^rG map $h : X \to Y$ in the C^r Whitney topology.

In the Nash case, if $1 \le r < \infty$, then locally C^r Nash diffeomorphisms are essentially different from C^r Nash diffeomorphisms because there exist two affine Nash manifolds such that they are C^∞ diffeomorphic but not Nash diffeomorphic [11], and that every C^r Nash diffeomorphism between affine Nash manifolds is approximated by a Nash diffeomorphism [12].

Proof of Proposition 3.2. Using the proof of [6] and Proposition 2.7, we may assume that Y is definably C^rG diffeomorphic to a definable C^rG submanifold closed in some representation Ω of G. By the similar way of finding a C^r partition of unity of C^r manifold, we have a locally definable C^r partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinates to some locally finite open definable cover $\{X_j\}_{j=1}^\infty$ of X such that $X = \cup_{j=1}^\infty \mathrm{supp}\ \phi_j$ and $\overline{X_j}$ is compact. For any j, take an open neighborhood U_j of supp ϕ_j in X such that $\overline{U_j}$ is compact. Applying the polynomial approximation theorem, we have a locally definable C^r map $h_j : U_j \to \Omega$ which approximates $f|U_j$. By Proposition 2.6, one can find a definable C^rG tubular neighborhood (U, p) of Y in Ω. If our approximation is sufficiently close, then $p \circ \sum_{j=1}^\infty \phi_j h_j$ is a (nonequivariant) C^r approximation of f. Since G is a finite group, applying Proposition 2.7, we have the required locally definable C^rG map h as a C^r Whitney approximation of f. □

Proof of Theorem 1.1. Let X be a definable C^rG manifold.

Let $\mathfrak{A}$ be the family of all pairs (A, α) such that each (A, α) consists of a non-empty open G invariant subset A of X and a locally definable C^s manifold structure α on A such that the action on $A_\alpha = (A, \alpha)$ is of class locally definable C^s and that the resulting locally definable C^sG manifold A_α is definable C^rG diffeomorphic to A with the original action.

We first show that $\mathfrak{A}$ is non-empty. Let $y \in X$ and let $K' = G_y$. By Theorem 2.10, there exists a linear definable C^r slice S' at y, in other words, there exists a definable C^rK imbedding i'' from a representation Ω of K into X such that $i''(\Omega) = S', i''(0) = y, A := GS'$ is open in X, and that

$$\mu(i'') : G \times_{K'} \Omega \to GS', \mu(i'')([g, x]) = gi''(x)$$

is a definable C^rG diffeomorphism. Since the twisted product $G \times_{K'} \Omega$ is a definable C^sG ($C^\omega G$) manifold, we can give A the definable C^s manifold structure α induced from $G \times_{K'} \Omega$ through $\mu(i'')^{-1}$. Since $\mu(i'')$ is a definable C^rG diffeomorphism, the resulting definable C^sG manifold A_α and the original definable C^rG manifold A are definably C^rG diffeomorphic. Hence $(A, \alpha) \in \mathfrak{A}$. Therefore $\mathfrak{A}$ is non-empty.

We give an order in $\mathfrak{A}$ by setting

$$(A_1, \alpha_1) \le (A_2, \alpha_2)$$

if and only if:

(1) $A_1 \subset A_2$.

(2) The locally definable C^sG manifold structure α_1 on A_1 is the one induce from that of α_2 on A_2.

Suppose that C is a subset of $\mathfrak{A}$ satisfying either $(A_1, \alpha_1) \le (A_2, \alpha_2)$ or $(A_1, \alpha_1) \ge (A_2, \alpha_2)$ if $(A_1, \alpha_1), (A_2, \alpha_2) \in C$.

Let C' be the family of all A occurring as the first coordinate of a pair in C, and let C'' denote the family of all α occurring as the second coordinate of a pair in C. We define

$$A^* = \cup_{A \in C'} A, \alpha' = \cup_{\alpha \in C''} \alpha$$

Then A^* is a non-empty open G invariant subset of X, and the resulting locally definable C^sG manifold $A^*_{\alpha'} = (A^*, \alpha')$ is definable C^rG diffeomorphic to A^* with the original definable C^rG action. Moreover it can be seen easily that $(A^*, \alpha') \in \mathfrak{A}$ and $(A, \alpha) \le (A^*, \alpha')$ for any $(A, \alpha) \in C$. Thus by Zorn's lemma, we have a maximal element $(A, \alpha) \in \mathfrak{A}$.

Remark that if C is finite and all A_α in C are definable C^sG manifold structures, then

$A^*_{\alpha'}$ is a definable C^sG manifold structure.

We prove that $A = X$ as follows.

Assume that $A \neq X$. If A is closed in X, then $X - A$ is a non-empty open G invariant subset of X. By the argument at the beginning of the proof, one can find a non-empty open G invariant subset B with $B \subset X - A$ such that B_β is a locally definable C^sG manifold which is definable C^rG diffeomorphic to the original structure B. Thus the locally definable C^sG manifold $(A \amalg B, \alpha \amalg \beta)$ defined by (A, α) and (B, β) is definably C^rG diffeomorphic to the disjoint union $A \amalg B$ with the original definable C^rG action, hence $(A \amalg B, \alpha \amalg \beta) > (A, \alpha)$. This contradicts the maximality of (A, α). Therefore A is not closed, and hence $\overline{A} - A \neq \emptyset$.

Let $x \in \overline{A} - A$ and let K denote G_x. By Theorem 2.10, there exists a linear definable slice S at x in X. Let i be a definable C^rK diffeomorphism from a representation Ξ of K onto $S \subset X$ such that $i(\Xi) = S$, $i(0) = x$, GS is open in X, and that

$$\mu(i) : G \times_K \Xi \to GS, \mu(i)([g, x]) = gi(x)$$

is a definable C^rG diffeomorphism.

Let l be a positive real number. Let $B_l^\circ := \{x \in \Xi | ||x|| < l\}$ $(\subset \Xi)$, $B_l := \{x \in \Xi | ||x|| \leq l\}$ $(\subset \Xi)$, $D_l^\circ := i(B_l^\circ)$ $(\subset S)$, and $D_l := i(B_l)$ $(\subset S)$, where $||x||$ denotes the standard norm of x. Since $x \in \overline{A} - A$ and GD_l° is an open neighborhood of x in X, we have

$$GD_l^\circ \cap A \neq \emptyset \text{ and } GD_l^\circ \cap (X - A) \neq \emptyset.$$

Moreover we obtain that $G(D_l^\circ \cap A) = GD_l^\circ \cap A$ because A is G invariant, and thus $D_l^\circ \cap A \neq \emptyset$. Hence $D_l^\circ \cap A$ is a non-empty open K invariant subset of D_l° because A is open in X. Then $V := i^{-1}(D_l^\circ \cap A)$ $(= B_l^\circ \cap i^{-1}(A))$ is a non-empty open K invariant subset of Ξ and $i(V) = D_l^\circ \cap A$. Shrinking S and modifying i, if necessary, we may assume that V is definable. Thus V is an affine definable C^sK manifold.

Since D_l° is open, $D_l^\circ \cap A_\alpha$ is a locally definable C^sK manifold.

Consider a definable C^rK diffeomorphism

$$i|V : V \to D_l^\circ \cap A_\alpha.$$

By Lemma 3.1, one can find an open neighborhood $\mathfrak{N}$ of $i|V$ in $Def_K^r(V, D_l^\circ \cap A_\alpha)$ such that the map $E : \mathfrak{N} \to Def_K^r(B_l^\circ, D_l^\circ)$ defined by

$$E(h)(x) = \begin{cases} h(x), & x \in V \\ i(x), & x \in B_l^\circ - V \end{cases}, h \in \mathfrak{N}$$

is continuous. Remark that $E(i|V) = i|B_l^\circ$. Since E is continuous, if we choose a sufficiently small $\mathfrak{N}$, each element of $E(\mathfrak{N})$ is a definable C^rK diffeomorphism.

Applying Proposition 3.2 to $(i|V)^{-1}$, we have a locally definable C^sK diffeomorphism $h_1 : V \to i(V)$ as a definable C^s approximation of $i|V$. If our approximation is sufficiently close, then $h_1 \in \mathfrak{N}$, and

$$j := E(h_1) : B_l^\circ \to D_l^\circ$$

is a definable C^sK diffeomorphism. Hence

$$\mu(j) : G \times_K B_l^\circ \to GD_l^\circ,$$
$$\mu(j)([g, x]) = gj(x)$$

is a definable C^sG diffeomorphism.

We claim that the restriction

$$\mu(j)|G \times_K V : G \times_K V \to GD_l^\circ \cap A_\alpha$$

is a locally definable C^sG diffeomorphism. Since $\operatorname{Im} i = \operatorname{Im} j|V$, $\mu(j)|G \times_K V$ is a definable C^sG diffeomorphism onto $GD_l^\circ \cap A_\alpha$, where $\operatorname{Im} i$ and $\operatorname{Im} j|V$ denote the image of i and $j|V$, respectively. Since $j|V = h_1 : V \to D_l^\circ \cap A_\alpha$ is a locally definable C^sK map and by Proposition 2.8, $\mu(j)|G \times_K V : G \times_K V \to GD_l^\circ \cap A_\alpha$ is a locally definable C^sG map. Thus $\mu(j)|G \times_K V : G \times_K V \to GD_l^\circ \cap A_\alpha$ is a definable C^sG diffeomorphism and of class locally definable C^r. Hence $\mu(j)|G \times_K V : G \times_K V \to GD_l^\circ \cap A_\alpha$ is a locally definable C^sG diffeomorphism.

Since $W := GD_l^\circ \subset X$ is open in X and G invariant, we can give W the locally definable C^sG manifold structure δ induced from $G \times_K B_l^\circ$ through $\mu(j)^{-1}$. Since $\mu(j) : G \times_K B_l^\circ \to W$ is a definable C^sG diffeomorphism, W_δ is definable C^sG diffeomorphic to W with the original action. Hence the induced locally definable C^sG manifold structures on $W \cap A$ from A_α and $\mu(j)^{-1}|W \cap A$ are the same. Thus $\alpha \cup \delta$ is a locally definable C^s atlas on $A \cup W$ such that the action on $(A \cup W)_{\alpha \cup \delta}$ is of class locally definable C^s and definably C^sG equivalent to the original action on $A \cup W$. Hence $(A \cup W)_{\alpha \cup \delta} \in \mathfrak{A}$. Moreover since $W \cap (X - A) \neq \emptyset$ and $A \neq A \cup W$, $(A, \alpha) < (A \cup W, \alpha \cup \delta)$. This contradicts the maximality of (A, α). Therefore $A = X$. $\square$

4. Proof of Theorem 1.2

In this section we prove the following theorem.

Theorem 4.1. *Let G be a finite group and let r be a positive integer. Suppose that $f : Y \to Z$ is a C^rG diffeomorphism between affine locally definable C^rG manifolds Y and Z. Then there exists a locally C^rG diffeomorphism $h : Y \to Z$ which is G homotopic to f.*

Remark that for any two $C^\infty G$ manifold imbedded into some representations of G, they are C^1G diffeomorphic if and only if they are $C^\infty G$ diffeomorphic. This $C^\infty G$ diffeomorphism is obtained by approximating the original C^1G diffeomorphism by a non-equivariant C^∞ diffeomorphism and by averaging this C^∞ diffeomorphism.

Thus Theorem 1.2 follows from Theorem 4.1.

For simplicity, we use the following notations. Let K be a subgroup of G and let X be a definable C^rG manifold. By Theorem 2.10, there exists a linear definable C^rK slice S, namely there exists a definable C^rK diffeomorphism i from some representation Ω of K to S such that GS is open in X, and that $\mu : G \times_K \Omega \to GS \ (\subset X), \mu(i)([g,x]) = gi(x)$ is a definable C^rG diffeomorphism.

Set $B_s := \{x \in \Omega | ||x|| \leq s\}, B_s^\circ := \{x \in \Omega | ||x|| < s\}, s > 0, B := B_1$, and $B^\circ := B_1^\circ$, and let denote D_s, D_s°, D and D° by $i(B_s), i(B_s^\circ), i(B)$, and $i(B^\circ)$, respectively. Let GD (resp. GD°) denote the closed unit tube (resp. the open unit tube), and let GD_s° stand for the open tube of radius s.

To prove Theorem 4.1 we prepare two preliminary results.

Lemma 4.2. *Let Ω and Ξ be representations of G and let M (resp. N) be a definable C^rG submanifold of Ω (resp. Ξ). Suppose that F is a G invariant definable subsets of M and that $\alpha : M \to N$ is a C^rG map such that $\alpha|F : F \to N$ is definable. Let $\mathfrak{N}$ be a neighborhood of α in the set $C_G^r(M,N)$ of C^rG maps from M to N and let V_1 and V_2 be compact G invariant definable subsets of M such that V_1 is properly contained in the interior Int V_2 of V_2. Then there exists $\kappa \in \mathfrak{N}$ such that:*
(a) $\kappa | F \cup V_1 : F \cup V_1 \to N$ is definable.
(b) $\kappa = \alpha$ on $M -$ Int V_2
(c) κ is G homotopic to α relative to $M -$ Int V_2

Proof. Take a non-negative definable C^r function $f : M \to \mathbb{R}$ such that $f = 0$ on V_1 and $f = 1$ on $M - \text{Int } V_2$. Since G is a finite group and by Proposition 2.7, we may assume that f is G invariant.

We approximate α by a polynomial G map β on V_2 using the polynomial approximation theorem and Proposition 2.7. By Proposition 2.6, one can find a definable C^rG tubular neighborhood (U, p) of N in Ξ. If the approximation is sufficiently close, then one can define

$$\kappa : M \to N,$$
$$\kappa(x) = p(f(x)\alpha(x) + (1 - f(x))\beta(x)).$$

Then κ is a C^rG map, and κ satisfies Properties (a) and (b). If this approximation is sufficiently close, then $\kappa \in \mathfrak{N}$ because κ and α coincide with outside of a compact set V_2.

The map $H : M \times [0,1] \to N$ defined by $H(x,t) = p((1-t)\alpha(x) + t\kappa(x))$ gives a G homotopy relative to $M - \text{Int } V_2$ from α to κ. $\square$

Proposition 4.3. *Let Ω and Ξ be representations of G. Let $M \subset \Omega$ and $N \subset \Xi$ be affine locally definable C^rG manifolds, A a closed G invariant locally definable subset of M and $0 < r < \infty$. Suppose that $f : M \to N$ is a C^rG diffeomorphism such that $f|A : A \to N$ is locally definable, and that $x \in M$. Suppose further that $j : \Omega' \to S$ is a linear definable C^r slice at*

x in Ω. If $GD_{10} \cap M$ is compact, then there exists a C^rG diffeomorphism $h : M \to N$ such that:
(1) $h|A \cup (GD \cap M) : A \cup (GD \cap M) \to N$ is locally definable.
(2) $h = f$ on $M - GD_2^\circ \cap M$.
(3) h is G homotopic to f relative to $M - GD_2^\circ \cap M$.

The condition that $GD_{10} \cap M$ is compact is not essential. By Theorem 2.10, one can find a linear definable C^r slice S at $x \in M$ in Ω. Since S is a linear definable C^r slice in Ω, there exists a definable C^rK diffeomorphism j from some representation Ω' of G_x onto S such that $j(0) = x$, GS is open in Ω, and that

$$\mu(j) : G \times_{G_x} \Omega' \to GS \ (\subset \Xi),$$
$$\mu(j)([g, x]) = gj(x)$$

is a definable C^rG diffeomorphism. Notice that M is locally compact. Thus replacing smaller S, if necessary, $GD_{10} \cap M$ is compact because M is locally compact.

Proof. Since $GD_{10} \cap M$ is compact and A is closed in M, $A \cap GD_{10}$ $(= A \cap (GD_{10} \cap M))$ is a compact G invariant locally definable subset of $GS \cap M$. Thus $A \cap GD_{10}$ is a G invariant definable subset of Ω. Hence

$$E := \mu(j)^{-1}(A \cap GD_{10})$$

is a G invariant definable subset of $G \times_{G_x} \Omega'$. Let $L = j^{-1}(D_{10}^\circ \cap M)$. The map

$$\alpha := f \circ \mu(j)|G \times_{G_x} L : G \times_{G_x} L \to \Xi$$

is a C^rG diffeomorphism onto an open G invariant subset $V := f(GD_{10}^\circ \cap M)$ of N. Since $A \cap GD_{10}$ is compact and $f|A$ is locally definable, $f|(A \cap GD_{10}) : A \cap GD_{10} \to f(A \cap GD_{10}) \subset N \subset \Xi$ is definable. The map $\alpha|(E \cap (G \times_{G_x} L)) : E \cap (G \times_{G_x} L) \to \Xi$ is definable because $\mu(j)$ and $f|(A \cap GD_{10}) : A \cap GD_{10} \to \Xi$ are definable. Since V is contained in a G invariant compact set $f(GD_{10} \cap M)$, and since N is a locally definable C^rG submanifold of Ξ, there exists a G invariant definable set W of Ξ such that $V \subset W \subset N$ and that W is open in N. Notice that W is an affine definable C^rG manifold. Since $G \times_{G_x} L$ is contained in a G invariant compact subset of $G \times_{G_x} j^{-1}(D_{20} \cap M)$, $G \times_{G_x} L$ is an affine definable C^rG manifold. Applying Lemma 4.2 to $\alpha : G \times_{G_x} L \to W$, there exists a C^rG map $\beta : G \times_{G_x} L \to W$ as a C^r Whitney approximation of α such that:
(a) $\beta|(G \times_{G_x} (j^{-1}(A \cap D_{10}^\circ) \cup (B \cap L))) : G \times_{G_x} (j^{-1}(A \cap D_{10}^\circ) \cup (B \cap L)) \to W$ $(\subset N)$ is definable.
(b) $\beta = \alpha$ on $G \times_{G_x} (L - B_2^\circ \cap L)$.
(c) β is G homotopic to α relative to $G \times_{G_x} (L - B_2^\circ \cap L)$.

Then the map $h : M \to N$ defined by

$$h(x) = \begin{cases} \beta \circ \mu(j)^{-1}(x), & x \in GD_5 \cap M \\ f(x), & x \in M - M \cap GD_5^\circ \end{cases}$$

is well-defined, and it is a C^rG diffeomorphism if our approximation is sufficiently close. Since $h|(A \cap GD_5)$ and $h|(GD \cap M)$ are definable, and since $h|(A \cap (M - GD_5 \cap M))$ $(= f|(A \cap (M - GD_5 \cap M)))$ is locally definable, $h|A \cup (GD \cap M)$ is locally definable by Proposition 2.1. By the construction of h, h satisfies Properties (2) and (3). □

Proof of Theorem 4.1. Let Ω be a representation of G including Y as a locally definable C^rG submanifold.

Recall that the algebra of G invariant polynomials on Ω is finitely generated [14]. Take its generators $p_1, \dots, p_l$ and let $p : \Omega \to \mathbb{R}^l, p = (p_1, \dots, p_l)$. Then p is a proper polynomial map. Moreover p induces a closed imbedding $F : \Omega/G \to \mathbb{R}^l$ such that $p = F \circ \pi'$, where π' denotes the orbit map $\Omega \to \Omega/G$. Thus we may identify Ω/G and π' with $p(\Omega)$ and p, respectively.

The orbit map $\pi : Y \to Y/G$ is the restriction of $\pi' : \Omega \to \Omega/G$. Hence Y/G is a locally definable subset of $\mathbb{R}^l$ because G is finite. Moreover π takes every locally definable subset of Y to a locally definable subset of Y/G. It follows from locally compactness and locally definablity of Y/G that there exists an expanding sequence $\{C_i\}_{i=0}^\infty$ of compact definable subsets of Y/G such that

$$C_0 \subset \text{Int } (C_1) \subset \dots \subset C_i \subset \text{Int } (C_{i+1})$$
$$\subset C_{i+1} \subset \dots,$$

and $Y/G = \cup_{i=0}^\infty C_i$.

Define $A_i := \pi^{-1}(C_i), i \geq 0$. Then each A_i ($\subset Y$) is a closed G invariant definable subset of Y and $\{A_i\}_{i=0}^{\infty}$ satisfies the following three conditions:
(a) A_i/G, $(i \geq 0)$ is compact.
(b) $A_0 \subset \text{Int}\,(A_1) \subset \cdots \subset A_i \subset \text{Int}\,(A_{i+1}) \subset A_{i+1} \subset \ldots$.
(c) $Y = \cup_{i=0}^{\infty} A_i$.

For each $x \in Y$ ($\subset \Omega$), there exists a linear definable C^r slice S at x in Ω by Theorem 2.10. Hence $\mu : G \times_{G_x} S \to GS$ ($\subset \Omega$) is a definable C^rG diffeomorphism. Hence if S is sufficiently small, then $S \cap Y$ and $GS \cap Y$ are definable and $\mu | G \times_{G_x} (S \cap Y) : G \times_{G_x} (S \cap Y) \to G(S \cap Y) = GS \cap Y$ ($\subset Y \subset \Omega$) is a definable C^rG diffeomorphism. Since S is linear, S is definable G_x diffeomorphic to some representation Ξ of G_x. Hence $GD^\circ \cap Y$ is an open G invariant definable subset of Y containing x. Thus we obtain an open cover $\{GD^\circ \cap Y\}_\alpha$ (resp. $\{\pi(GD^\circ \cap Y)\}_\alpha$) of Y (resp. Y/G) consisting of open G invariant definable subsets of Y (resp. open definable subsets of Y/G).

The rest of the proof of our theorem is divided into two steps.

In the first step, we now construct a C^rG diffeomorphism $\hat{f} : Y \to Z$ such that $\hat{f}$ is G homotopic to f and that $\hat{f}|\hat{E} : \hat{E} \to Z$ is locally definable, where $\hat{E}$ is a closed G invariant locally definable subset of Y and

$$\hat{E} \supset \cup_{n=0}^{\infty}(A_{4n} - \text{Int}\,(A_{4n-2})).$$

Thus $\hat{E}$ contains "half" of Y under the filtration of $A_i, i \geq 0$, of Y. Let

$$E_n = A_{4n} - \text{Int}\,(A_{4n-2}),$$

$$U_n = \text{Int}\,(A_{4n+1}) - A_{4n-3}, n \geq 0,$$

where $A_{-3} = A_{-2} = A_{-1} = \emptyset$. Hence E_n is a closed G invariant definable subset of Y and U_n is an open definable neighborhood of E_n in Y, and

$$U_n \cap U_{n'} = \emptyset \text{ for } n \neq n'.$$

Notice that

$$Y = \cup_{n=0}^{\infty} \overline{U_n}.$$

By induction, we now construct C^rG diffeomorphisms $f_n : Y \to Z, n \geq 0$ and closed G invariant locally definable subsets E_n^* of Y such that:
(a) $E_n \subset E_n^* \subset U_n, n \geq 0$
(b) $f_n|E_0^* \cup \cdots \cup E_n^* : E_0^* \cup \cdots \cup E_n^* \to Z$ is locally definable.
(c) $f_n = f_{n-1}$ on $Y - U_n$.
(d) f_n is G homotopic to f_{n-1} relative to $Y - U_n$.

Suppose that f_n and E_n^* have already been constructed which satisfy the above conditions, where $f_{-1} = f, E_{-1}^* = \emptyset$ and $n \geq -1$.

Then we now construct f_{n+1} and E_{n+1}^* in the following way. Since E_{n+1}/G is compact and U_{n+1} is an open G invariant definable neighborhood of E_{n+1}, and by Proposition 4.3, there exist a finite number of closed unit tubes $GD^1, \ldots, GD^k$ such that

$$E_{n+1} \subset \cup_{j=1}^{k}(GD^j \cap Y) \subset \cup_{j=1}^{k}(GS_j \cap Y) \subset U_{n+1},$$

and that there exist C^rG diffeomorphisms

$$f_{n,j} : Y \to Z, 1 \leq j \leq k$$

such that:
(a) $f_{n,j}|(\cup_{i=0}^{n}E_i^*) \cup (GD^1 \cap Y) \cup \cdots \cup (GD^j \cap Y) : (\cup_{i=0}^{n}E_i^*) \cup (GD^1 \cap Y) \cup \ldots \cup (GD^j \cap Y) \to Z$ is locally definable.
(b) $f_{n,j} = f_{n,j-1}$ on $Y - (G(D^j)_2^\circ \cap Y)$
(c) $f_{n,j}$ is G homotopic to $f_{n,j-1}$ relative to $Y - (G(D^j)_2^\circ \cap Y)$, where $f_{n,0} = f_{n-1}$.

Set $f_{n+1} := f_{n,k}$ and $E_{n+1}^* := \cup_{j=1}^{k}(GD^j \cap Y)$. Then

$$f_{n+1}|E_0^* \cup \cdots \cup E_{n+1}^* : E_0^* \cup \cdots \cup E_{n+1}^* \to Z$$

is locally definable. Since $G(D^j)_2^\circ \cap Y \subset GS_j \cap Y \subset U_{n+1}$ for $1 \leq j \leq k$, it follows that $f_{n+1} = f_n$ on $Y - U_{n+1}$, and that f_{n+1} is G homotopic to f_n relative to $Y - U_{n+1}$. Hence f_n and E_n^* are required ones.

Now $f_{n+1} = f_n$ on A_{4n+1}, and hence

$$\hat{f} : Y \to Z, \hat{f}(x) = f_n(x) \text{ if } x \in A_{4n+1}$$

is a well-defined map. Clearly $\hat{f}$ is an injective C^rG immersion.

We now prove that $\hat{f} : Y \to Z$ is surjective. We have

$$f_n|(Y - U_n) = f_{n-1}|(Y - U_n), n \geq 0,$$

and since both f_n and f_{n-1} are bijective, this implies that

$$f_n(U_n) = f_{n-1}(U_n).$$

Moreover we have that $U_i \subset Y - U_n$ and $f_n(U_i) = f_{n-1}(U_i)$ for all $i \neq n$. Thus and $f_n(U_i) = f_{n-1}(U_i)$ for $i \geq 0$, and since f_n, f_{n-1} are homeomorphisms,

$$f_n(\overline{U_i}) = f_{n-1}(\overline{U_i}) \text{ for } i \geq 0.$$

Therefore $f(\overline{U_i}) = \hat{f}(\overline{U_i})$ for $i \geq 0$,

$$\hat{f}(Y) = \cup_{i=0}^{\infty}\hat{f}(\overline{U_i}) = \cup_{i=0}^{\infty} f(\overline{U_i}) = f(Y) = Z.$$

Hence $\hat{f}$ is a bijective C^rG immersion, and thus by the inverse function theorem, $\hat{f} : Y \to Z$ is a C^rG diffeomorphism.

Since $E_n^* \subset U_n \subset A_{4n+1}$, we have that $\hat{f}|E_n^* = f_n|E_n^*$, and hence $\hat{f}|E_n^* : E_n^* \to Z$ is locally definable. On the other hand, $\{E_n^*\}_{n\geq 0}$ is a locally finite family of closed locally definable subsets of Y. Thus

$$\hat{f} : \hat{E} \to Z$$

is locally definable by Proposition 2.1, where

$$\hat{E} = \cup_{n=0}^{\infty} E_n^*.$$

Then $\hat{E}$ is closed, and $\hat{E}$ satisfies $\hat{E} = \cup_{n=0}^{\infty}(A_{4n} - \text{Int } A_{4n-2})$ because $E_n \subset E_n^*$, $n \geq 0$.

Since f_{n-1} is G homotopic to f_n relative to A_{4n-3}, $n \geq 0$, a G homotopy from f to $\hat{f}$ is constructed by a standard procedure (See. e.g. the proof of Theorem 7.6.17 [13]).

In the second step, we now produce a C^rG diffeomorphism $h : Y \to Z$ such that h is G homotopic to $\hat{f}$ and that $h|\hat{E} \cup \hat{F} : \hat{E} \cup \hat{F} \to Z$ is locally definable by the same way as above, where $\hat{F}$ is a closed G invariant locally definable subset of Y such that

$$\hat{F} \supset \cup_{n=0}^{\infty}(A_{4n+2} - \text{Int } (A_{4n})).$$

Then $Y = \hat{E} \cup \hat{F}$, and hence $h : Y \to Z$ is a locally definable C^rG diffeomorphism.

Let $n \geq 0$. Set

$$\begin{cases} F_n = A_{4n+2} - \text{Int } (A_{4n}) \\ V_n = \text{Int } (A_{4n+3}) - A_{4n-1} \end{cases}.$$

Now we construct a C^rG diffeomorphism $h_n : Y \to Z, n \geq 0$ and closed G invariant locally definable subsets F_n^* of Y such that:

(a) $F_n \subset F_n^* \subset V_n, n \geq 0$.

(b) $h_n|\hat{E} \cup F_0^* \cup \cdots \cup F_n^* : \hat{E} \cup F_0^* \cup \cdots \cup F_n^* \to Y$ is locally definable.

(c) $h_n = h_{n-1}$ on $Y - V_n$.

(d) h_n is G homotopic to h_{n-1} relative to $Y - V_n$,

where $h_{-1} = \hat{f}$.

This construction is accomplished inductively by finitely many use of Proposition 4.3 as in the first part of the proof. Since $h_n = h_{n+1}$ on A_{4n+3}, we get a well-defined map

$$h : Y \to Z, h(x) = h_n(x) \text{ if } x \in A_{4n+3}.$$

Then $h : Y \to Z$ is an injective C^rG immersion. Moreover we can show that h is surjective in the same way as we proved that $\hat{f} : Y \to Z$ is surjective. Hence h is a C^rG diffeomorphism by the inverse function theorem.

Since $E_n^* \cup F_n^* \subset A_{4n+3}$, we obtain that $h|E_n^* \cup F_n^* = h_n|E_n^* \cup F_n^*$, and hence $h|E_n^* \cup F_n^* : E_n^* \cup F_n^* \to Z$ is locally definable. Now $\{E_n^* \cup F_n^*\}_{n\geq 0}$ is a locally finite family of closed locally definable subsets of Y, and by Proposition 2.1, we have

$$h|\hat{E} \cup \hat{F} : \hat{E} \cup \hat{F} \to Z$$

is locally definable, where $\hat{F} = \cup_{n=0}^{\infty} F_n^*$. Since $F_n \subset F_n^*, n \geq 0$, $\hat{F}$ satisfies $\hat{F} = \cup_{n=0}^{\infty}(A_{4n+2} - \text{Int } A_{4n})$. Then $\hat{E} \cup \hat{F} = Y$, and thus $h : Y \to Z$ is of class locally definable C^r.

Since h_{n-1} is G homotopic to h_n relative to A_{4n-1} for $n \geq 0$, the same standard procedure that we already referred to give a G homotopy from $\hat{f}$ to h. Therefore h is the required diffeomorphism because h is G homotopic to f. □

5. Proof of Theorem 1.3

A similar way of the proof of Proposition 3.2, we have the following.

Proposition 5.1. *Let X be a locally definable C^r manifold and $1 \leq r < \infty$. Then every C^r map $f : X \to \mathbb{R}^n$ is approximated in the C^r Whitney topology by a locally definable C^r map $h : X \to \mathbb{R}^n$.*

Proof of Theorem 1.3. By Whitney's imbedding Theorem (e.g. 2.14 [1]), there exists a C^r imbedding $f : X \to \mathbb{R}^{2n+1}$. By Proposition 5.1 and since imbeddings from X to $\mathbb{R}^{2n+1}$ are open in $C^r(X, \mathbb{R}^{2n+1})$, we have the required a locally definable C^r imbedding $h : X \to \mathbb{R}^{2n+1}$. □

References

[1] M.W. Hirsch *Differential topology*, Springer-Verlag (1976).

[2] S. Illman, *Every proper smooth actions of a Lie group is equivalent to a real analytic action: a contribution to Hilbert's fifth problem*, Ann. Math. Stud. **138** (1995), 189-220.

[3] S. Illman, *Smoothly equivalent real analytic proper G manifolds are subanalytically equivalent*, Math. Ann. **306** (1996), 647-673.

[4] T. Kawakami, *Affineness of definable C^r manifolds and its applications*, Bull. Korean Math. Soc. **40**, (2003), 149-157.

[5] T. Kawakami, *Equivariant definable C^r approximation theorem, definable C^rG triviality of definable C^rG maps and Nash G compactifications*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. No. 55, (2005), 23-36

[6] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology Appl. **123** (2002), 323-349.

[7] T. Kawakami, *Every definable C^r manifold is affine*, Bull. Korean Math. Soc. **42** (2005), 165-167.

[8] T. Kawakami, *Imbedding of manifolds defined on an o-minimal structures on* $(\mathbb{R}, +, \cdot, <)$, Bull. Korean Math. Soc. **36** (1999), 183-201.

[9] T. Kawakami, *Nash G manifold structures of compact or compactifiable $C^\infty G$ manifolds*, J. Math. Soc. Japan **48** (1996), 321-331.

[10] F. Kutzschebauch, *On the uniqueness of the real analytic G-structure on a smooth proper G-manifold for G a Lie group*, Manuscripta Math. **90** (1996), 17-22.

[11] M. Shiota, *Abstract Nash manifolds*, Proc. Amer. Math. Soc. **96** (1986), 155-162.

[12] M. Shiota, *Approximation theorems for Nash mappings and Nash manifolds*, Trans. Amer. Math. Soc. **293** (1986), 319-337.

[13] E.H. Spanier, *Algebraic topology* McGraw-Hill New York (1966).

[14] H. Weyl, *The classical groups*, 2nd ed. Princeton Univ. Press, Princeton, N.J (1946).

Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani Wakayama 640-8510, Japan

E-mail address: kawa@center.wakayama-u.ac.jp