Some open problems in o-minimal expansions of the field of real numbers

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Abstract

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, >)$ of the field $\mathbb{R}$ of real numbers. In this paper, we collect some open problems in $\mathcal{M}$.

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1 . C^r Nash functions

A *semialgebraic subset* of $\mathbb{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^n | f_1(x) = \cdots = f_k(x) = 0, h_1(x) > 0, \dots, h_l(x) > 0\},$$

where $f_1, \dots, f_k, h_1, \dots, h_l \in \mathbb{R}[x_1, \dots, x_n]$. A *semialgebraic set* means a semialgebraic subset of some $\mathbb{R}^n$. A continuous map between semialgebraic sets is called *semialgebraic* if the graph of it is a semialgebraic set.

Semialgebraic sets and semialgebraic maps have satisfactory properties, for example, cell decompositions, triangulations, trivializations (e.g. [1]).

Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be semialgebraic open sets. A semialgebraic map $f : X \to Y$ is a C^r *Nash map* if f is a C^r map. A C^r Nash map $h : U \to V$ is called a C^r *Nash diffeomorphism* (a *semialgebraic homeomorphism* if $r = 0$) if there exists a C^r Nash map $k : V \to U$ such that $h \circ k = id$ and $k \circ h = id$.

Theorem 1.1. *Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be semialgebraic open sets.*

(1) ([11]) Every C^∞ Nash map $f : U \to V$ is a C^ω Nash map.

(2) (e.g. [1]) The set $N^\omega(U)$ of C^ω Nash functions on U is Noetherian.

2 . Definable C^r functions

Recall the definition of *structures* from model theory.

A *structure* $\mathcal{M}$ is given by the following data.

1. A set M called the *universe* or the *underlying set* of $\mathcal{M}$.

2. A collection of *functions* $\{f_i | i \in I\}$, where $f_i : M^{n_i} \to M$ for some $n_i \geq 1$.

3. A collection of *relations* $\{R_j | j \in J\}$, where $R_j \subset M^{m_j}$ for some $m_j \geq 1$.

4. A collection of distinguished elements $\{c_k | k \in K\} \subset M$, and each c_k is called a *constant*.

Any (or all) of the sets I, J, K may be empty. We refer n_i and m_j as the arity of f_i and R_j.

For simplicity, we only consider an o-minimal expansion $\mathcal{M}$ of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers.

We say that f (resp. R) is an *m-place function symbol* (resp. an *m-place relation symbol*) if $f : \mathbb{R}^m \to \mathbb{R}$ is a function (resp. $R \subset \mathbb{R}^m$ is a relation).

A *term* is a finite string of symbols obtained by repeated applications of the following two rules:

1. Variables are terms.

2. If f is an m-place function symbol of $\mathcal{M}$ and $t_1, \dots, t_m$ are terms, then the concatenated string $f(t_1, \dots, t_m)$ is a term.

Note that if $m = 0$, then the second rule says that constant symbols (0-place function symbols) are terms.

A *formula* is a finite string of symbols $s_1 \dots s_k$, where each s_i is either a variable, a function symbol, a relation symbol, one of the logical symbols $=, \neg, \vee, \wedge, \exists, \forall$, one of the brackets (,), or comma ,. Arbitrary formulas are generated inductively by the following three rules:

1. For any two terms t_1 and t_2, $t_1 = t_2$ and $t_1 > t_2$ are formulas.

2. If R is an m-place relation symbol and $t_1, \dots, t_m$ are terms, then $R(t_1, \dots, t_m)$ is a formula.

3. If ϕ and ψ are formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are formulas. If ϕ is a formula and v is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

A subset X of $\mathbb{R}^n$ is *definable* (in $\mathcal{M}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and elements $b_1, \dots, b_m \in \mathbb{R}$ such that $X = \{(a_1, \dots, a_n) \in \mathbb{R}^n | \phi(a_1, \dots, a_n, b_1, \dots, b_m)$ is true in $\mathcal{M}\}$.

Let $K \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^m$ be definable sets. We say that a continuous map $f : K \to L$ is *definable* (in $\mathcal{M}$) if the graph of f ($\subset K \times L \subset \mathbb{R}^n \times \mathbb{R}^m$) is definable. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be definable open sets. A C^r map $f : U \to V$ is called a *definable C^r map* if it is definable. A definable C^r map $h : U \to V$ is called a *definable C^r diffeomorphism* (a *definable homeomorphism* if $r = 0$) if there exists a definable C^r map $k : V \to U$ such that $h \circ k = id$ and $k \circ h = id$.

An *open interval* means something of the form $(a, b), a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{\infty\}$. We call $\mathcal{M}$ *o-minimal* (*order minimal*) if every definable subset of $\mathbb{R}$ is a finite union of points and open intervals. Remark that $\mathcal{R}$ is o-minimal. For example, $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \mathbb{Z})$ is an expansion of $\mathcal{R}$ but not o-minimal because a definable subset $\mathbb{Z}$ of $\mathbb{R}$ in $\mathcal{M}$ is not a finite union of points and open intervals.

Notice that one can consider a definable category in a structure which is not o-minimal. But this category does not have satisfactory properties. Notice further that one can define o-minimal structures over a non-empty set R with an order $<$, but one needs an addition and a multiplication to consider differential manifolds on $(R, <)$.

If $\mathcal{M} = \mathcal{R}$ $(= (\mathbb{R}, +, \cdot, <))$, then every definable set is a semialgebraic set [16], and a definable map is a semialgebraic map [16]. In particular, the semialgebraic category is a special case of the definable one.

Typical o-minimal structures on $\mathcal{R}$ are the following.

(1) $\mathbf{R}_{exp} := (\mathbb{R}, +, \cdot, <, exp)$, where exp denotes the exponential function $x \mapsto e^x$.

(2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where f ranges over all maps $f : \mathbb{R}^n \to \mathbb{R}, n \in \mathbb{N}$ such that $f|[-1, 1]^n$ is analytic and 0 outside of it.

(3) $\mathbf{R}_{an}^S := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where f is the same in (2) and $S \subset \mathbb{R}$ and

$x^r : \mathbb{R} \to \mathbb{R}, r \in S$ means

$$a \mapsto \begin{cases} a^r, & a > 0 \\ 0, & a \leq 0 \end{cases}.$$

(4) $\mathbf{R}_{an,exp} := (\mathbb{R}, +, \cdot, <, (f), exp)$, where (f) and exp are mentioned above.

For example, $\{y = e^x\} \subset \mathbb{R}^2$ and $\{y = x^l, x > 0, l \in \mathbb{R} - \mathbb{Q}\}$ are definable in $\mathbf{R}_{exp}$ but not in $\mathcal{R}$ (is not a semialgebraic set).

There are uncountable many o-minimal structures in $\mathcal{R}$ by [13].

An o-minimal expansion on $\mathcal{M}$ on $(\mathbb{R}, +, \cdot, <)$ is *polynomially bounded* if for any function $f : \mathbb{R} \to \mathbb{R}$ definable in $\mathcal{M}$, there exist an integer l and a real number x_0 such that $|f(x)| \leq x^l$ for all $x > x_0$. Otherwise, $\mathcal{M}$ is called *exponential*. If $\mathcal{M}$ is exponential, then the exponential function is a definable function in $\mathcal{M}$ [12].

If $\mathcal{M}$ is exponential, then $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x}}, x > 0 \\ 0, x \leq 0 \end{cases}$ is a definable C^∞ function but not a C^ω function.

Problem 2.1. *Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be definable open sets.*

(1) Let $\mathcal{M}$ be polynomially bounded. Is a definable C^∞ map $f : U \to V$ a definable C^ω map?

(2) Is the set of definable C^ω (resp. C^∞) functions on U Noetherian?

By [4], if $\mathcal{M} = \mathbb{R}_{an}$ and $n = 2$, then the set of definable C^ω functions on U is Noetherian.

3 . C^r Nash manifolds and definable C^r manifolds

Definition 3.1. *A Hausdorff space X with countable basis is a d-dimensional C^r Nash manifold if there exist a finite open covering $\{U_i\}_{i=1}^k$ of X and homeomorphisms ϕ_i from U_i to open subsets V_i of $\mathbb{R}^d$ $(1 \leq i \leq k)$ such that:*

(1) For all i, j, $\phi_i(U_i \cap U_j)$ is semialgebraic and open.

(2) For all i, j, $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a C^r Nash diffeomorphism.

Theorem 3.2 ([14])**.** *If $0 \leq r < \infty$, then every C^r Nash manifold is C^r Nash imbeddable into some $\mathbb{R}^n$. Namely, every C^r Nash manifold is affine.*

Theorem 3.3 ([14])**.** *Let X be a positive dimensional compact C^∞ manifold. Then there exists an uncountable family $\{Y_\lambda\}$ of nonaffine Nash manifolds such that for any λ, Y_λ is C^∞ diffeomorphic to X and that if $\mu \neq \lambda$, then Y_μ is not Nash diffeomorphic to Y_λ.*

Definition 3.4. *A Hausdorff space X with countable basis is a d-dimensional definable C^r manifold if there exist a finite open covering $\{U_i\}_{i=1}^k$ of X and homeomorphisms ϕ_i from U_i to open subsets V_i of $\mathbb{R}^d$ $(1 \leq i \leq k)$ such that:*

(1) For all i, j, $\phi_i(U_i \cap U_j)$ is definable and open.

(2) For all i, j, $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism.

Problem 3.5. *(1) Let r be a non-negative integer. Is a definable C^r manifold affine?*

(2) Is a definable C^∞ (resp. C^ω) manifold affine? How many nonaffine definable C^∞ (resp. C^ω) manifolds does there exist?

(3) How about equivariant versions of (1) and (2)?

Theorem 3.6. *([8]) (1) If $0 \leq r < \infty$, then every definable C^r manifold is affine.*

(2) If $2 \leq r < \infty$, then every n-dimensional definable C^r manifold is definably C^r imbeddable into $\mathbb{R}^{2n+1}$.

The second statement in Theorem 3.6 is the definable version of Whitney's imbedding theorem.

A group G is a *Nash group* if G is a Nash manifold and the group operations $G \times G \to G$ and $G \to G$ are Nash maps. A Nash group is *affine* if G is affine as a Nash manifold. A *Nash G manifold* is a pair (X, ϕ) consisting of a Nash manifold X and a group action $\phi : G \times X$ which is a Nash map. We simply write X instead of (X, ϕ). We call a Nash G manifold X *affine* if X is Nash

G imbeddable into some a representation of G.

Let G be a Lie group. A $C^\infty G$ *manifold* means a pair (X, ϕ) consisting a C^∞ manifold and a group operation $\phi : G \times X \to X$ which is a C^∞ map. We simply write X instead of (X, ϕ).

Theorem 3.7. *([9]) Let G be a compact affine Nash group. For every positive dimensional compact $C^\infty G$ manifold X without transitive action admits an uncountable family $\{Y_\lambda\}$ of nonaffine Nash G manifolds such that each Y_λ is $C^\infty G$ diffeomorphic to X and that Y_λ is not Nash G diffeomorphic to Y_μ for $\lambda \neq \mu$.*

Theorem 3.8. *(1) ([11]) Every C^∞ Nash map between Nash manifolds is a C^ω Nash map.*

(2) ([1]) The set of C^ω Nash functions on an affine Nash manifold is Noetherian.

Problem 3.9. *Let X, Y be definable C^ω manifolds.*

(1) Is a definable C^∞ map from X to Y a definable C^ω map?

(2) Is the set of definable C^ω (C^∞) functions on X Noetherian?

By [4], if $\mathcal{M} = \mathbb{R}_{an}$ and $\dim X = 2$, then the set of definable C^ω functions on X is Noetherian.

4. Three main properties of definable sets and maps

A point in $\mathbb{R}$ is a *definable C^r cell* and an open interval $(a, b), a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{\infty\}$ is a *definable C^r cell*. All *definable C^r cells* are defined inductively. For any definable C^r functions f, h on a definable C^r cell C, the graph of f, $\{(x, y) \in C \times \mathbb{R} | y < h(x)\}$ and $\{(x, y) \in C \times \mathbb{R} | f(x) < y\}$ are definable C^r cells. If $f(x) < h(x)$ for all $x \in C$, then $\{(x, y) \in C \times \mathbb{R} | f(x) < y < h(x)\}$ is a definable C^r cell. A *decomposition* of $\mathbb{R}$ is a collection $\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, \infty), \{a_1\}, \dots, \{a_k\}\}$, where $a_1 < \cdots < a_k$. A *decomposition* of $\mathbb{R}^n$ is a finite partition $\mathfrak{C}$ of $\mathbb{R}^n$ into definable C^r cells such that the set of projections $\pi(\mathfrak{C})$ is a decomposition of $\mathbb{R}^{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the usual projection. A decomposition $\mathfrak{D}$ of $\mathbb{R}^n$ into definable C^r cells is to *partition* a definable set $S \subset \mathbb{R}^n$ if each definable C^r cell in $\mathfrak{D}$ is either part of S or disjoint from S.

Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be definable sets. A continuous map $f : U \to V$ is a *definable C^r map* if there exist definable open sets $U' \supset U, V' \supset V$ and a definable C^r map $F : U' \to V'$ such that $F|U = f$.

Theorem 4.1. *(e.g. [3]) (Definable C^r cell decomposition). Let $S \subset \mathbb{R}^n$ be definable, $f : S \to \mathbb{R}$ a definable function. Then there exists a finite decomposition of $\mathbb{R}^n$ into definable C^r cells such that for any cell, $C \subset S$ or $C \subset \mathbb{R}^n - S$ and for any cell with $C \subset S$, $f|C : C \to \mathbb{R}$ is a definable C^r function.*

Theorem 4.2. *(e.g. [3])(Definable triangulation). Let $S_1, \dots, S_k$ be definable subsets of a definable set S in $\mathbb{R}^n$. Then there exist a finite simplicial complex $K \subset \mathbb{R}^n$ and a definable map $\phi : S \to \mathbb{R}^n$ such that ϕ maps S and each S_i homeomorphically onto unions of open simplexes of K.*

We call (ϕ, K) a *definable triangulation* of $S, S_1, \dots S_k$

Theorem 4.3. *(e.g. [3]) (Piecewise triviality) Let $f : S \to A$ be a definable map. Then there exist a finite partition of S into definable sets $\{C_i\}$ and definable maps $h_i : f^{-1}(C_i) \to f^{-1}(y_i)$ such that each $(f, h_i) : f^{-1}(C_i) \to C_i \times f^{-1}(y_i)$ is a definable homeomorphism, where $y_i \in C_i$.*

Problem 4.4. *(1) How about definable triangulations with group actions?*

(2) How about the C^r, the equivariant and the equivariant C^r versions of the Piecewise Triviality?

(3) When is it true the triviality instead of the piecewise triviality?

A group G is a *definable group* if G is a definable set and the group operations

$G \times G \to G$ and $G \to G$ are definable. A *representation map* of a definable group G is a group homomorphism from G to some $O(n)$ which is definable. A *representation* of G is the representation space of a representation map of G. A *definable G set* means a G invariant definable subset of some representation of G.

Let X be a definable G set. A definable triangulation (L, ϕ) of the orbit space X/G is *compatible with the orbit types* if for any orbit type (H), $\phi \circ \pi(X(H))$ is a union of open simplexes of L, where $\pi : X \to X/G$ denotes the orbit map and $X(H) = \{x \in X | (G_x) = (H)\}$.

Let G be a finite group. A *simplicial G complex* consists of a simplicial complex K together with a G action $\psi : G \times K \to K$ such that $\psi_g = \psi(g, \cdot) : K \to K$ is a simplicial homeomorphism for any $g \in G$.

We say that a simplicial G complex is an *equivariant simplicial complex* if the following two conditions are satisfied.

(1) For any subgroup H of G, if $\Delta^n =< v_0, \dots, v_n >$ and $\Delta^{n'} =< h_0 v_0, \dots, h_n v_n >$ are simplexes of K for $h_i \in H$, then there exists an $h \in H$ such that $h v_i = h_i v_i$ for all i.

(2) For every simplex Δ^n of K, the vertices $v_0, \dots, v_n$ of Δ^n can be ordered with $G_{v_n} \subset \dots \subset G_{v_0}$.

Proposition 4.5. *([10]) The second barycentric subdivision of any simplicial G complex is an equivariant simplicial complex.*

Let G be a finite group and X a definable G set. An *equivariant definable triangulation* (L, ϕ) of X consists of a G invariant union L of open simplexes of an equivariant simplicial complex and a definable G homeomorphism $\phi : |L| \to X$.

Theorem 4.6. *([6]) Let G be a finite group, X a definable G set in a representation Ω of G and r a positive integer. Then there exists an equivariant definable triangulation (L, ϕ) of X such that:*

(1) For any open simplex $int(\Delta^n)$ of L, $\phi(int(\Delta^n))$ is a locally closed definable C^r submanifold of Ω and $\phi | int(\Delta^n)$ is a definable C^r diffeomorphism onto its image.

(2) This triangulation induces a definable triangulation of X/G compatible with the orbit types.

In particular, if X is compact, then we can take L to be an equivariant simplicial complex.

Let X, Y be definable G sets. A definable G map $f : X \to Y$ is *definably G trivial* if there exist $y \in Y$ and a definable G map $h : X \to f^{-1}(y)$ such that $(f, h) : X \to Y \times f^{-1}(y)$ is a definable G homeomorphism.

Theorem 4.7. *([7]) Let G be a compact definable group and let S be a definable G set in some representation Ω of G. Let A be a definable set in some $\mathbb{R}^n$ and let $f : S \to A$ be a G invariant definable map. Then there exists a finite partition $\{A_i\}$ of A into definable sets such that each $f | f^{-1}(A_i) : f^{-1}(A_i) \to A_i$ is definably G trivial.*

The projection onto S^n of the tangent bundle of the standard n-dimensional sphere S^n with the standard $O(n+1)$ action for $n \geq 8$ is not piecewise definably trivial because the action is transitive and this bundle is not trivial. This example shows that we cannot drop the G invariant condition in Theorem 4.7.

A group G is a *definable C^r group* if G is a definable C^r manifold and the group operations $G \times G \to G$ and $G \to G$ are definable C^r maps. A definable C^r group is *affine* if G is affine as a definable C^r manifold.

Let G be a definable C^r group. A *definable $C^r G$ manifold* is a pair (X, θ) consisting of a definable C^r manifold X and a group action θ of G on X such that $\theta : G \times X \to X$ is a definable C^r map. For simplicity of notation, we write X instead of (X, θ).

Theorem 4.8. *([7]) Let G be a compact definable C^r group and $1 \leq r < \infty$. Let*

S be a definable C^rG submanifold of a representation of G and let A be a definable C^r submanifold of $\mathbb{R}^n$. For any G invariant surjective submersive definable C^r map $f: S \to A$, there exists a finite partition $\{A_i\}$ of A into definable C^r submanifolds such that each $f|f^{-1}(A_i) : f^{-1}(A_i) \to A_i$ is definably C^rG trivial. Moreover we can take $r = \omega$ (resp. $r = \infty$) if $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition.

A map $f : X \to Y$ between topological spaces is called *proper* if for any compact subset C of Y, $f^{-1}(C)$ is compact.

Theorem 4.9. *([2]) Let X be an affine Nash manifold. Then every surjective proper Nash submersion $f : X \to \mathbb{R}$ is Nash trivial.*

Problem 4.10. *(1) How about the equivariant version of Theorem 4.9?*

(2) How about the definable C^r version of it?

Theorem 4.11. *([5]) Let X be an affine definable C^rG manifold and $1 \leq r < \infty$. Then every G invariant proper submersive surjective definable C^r function $f : X \to \mathbb{R}$ is definable C^rG trivial.*

As a corollary of Theorem 4.11, we have the equivariant version of Theorem 4.9 when G is a finite group.

Corollary 4.12. *Let G be a finite group and X an affine Nash G manifold. Then any G invariant proper surjective Nash submersion $f : X \to \mathbb{R}$ is Nash G trivial.*

5. Compactifiable G manifolds

Let G be a Lie group. A non-compact $C^\infty G$ manifold X is *compactifiable* as a $C^\infty G$ manifold if it is $C^\infty G$ diffeomorphic to the interior of some compact $C^\infty G$ manifold with boundary.

Theorem 5.1. *([15]) Every non-compact affine Nash manifold is compactifiable as a C^∞ manifold.*

Problem 5.2. *(1) Let G be a compact Nash group. Is any non-compact Nash G manifold compactifiable as a $C^\infty G$ manifold?*

(2) Let G be a definable C^r group and $1 \leq r \leq \omega$. Is any non-compact definable C^rG manifold compactifiable as a definable C^rG manifold?

Theorem 5.3. *([9]) Let G be a compact affine Nash group.*

(1) Every affine Nash G manifold is compactifiable as a $C^\infty G$ manifold.

(2) A $C^\infty G$ manifold is $C^\infty G$ diffeomorphic to some affine Nash G manifold if and only if it is compactifiable as a $C^\infty G$ manifold.

Let G be a compact definable C^r group. A non-compact definable C^rG manifold is *compactifiable* as a definable C^rG manifold if it is definably C^rG diffeomorphic to the interior of some compact definable C^rG manifold with boundary.

Theorem 5.4. *([7]) Let G be a compact definable C^r group and $1 \leq r < \infty$. Every affine definable C^rG manifold is compactifiable as a definable C^rG manifold.*

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