Definable relative covering homotopy theorem and covering mapping cylinder conjecture

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Abstract

We prove the definable relative version of the Palais' covering homotopy theorem. Moreover we prove affirmatively the Bredon's covering mapping cylinder conjecture.

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1 . Introduction

R.S. Palais proved in [12] the following covering homotopy theorem.

Theorem 1.1 ([12] or II.7.3 [1]). *Let G be a compact Lie group and X, Y G spaces. Suppose that every open subspace of X/G is paracompact and $f : X \to Y$ is a G map with the induced map $f' : X/G \to Y/G$ between the orbit spaces. Let $F' : X \times [0,1] \to Y/G$ be an orbit structure preserving homotopy of f'. Then there exists a G homotopy $F : X \times [0,1] \to Y$ of f which covers F', namely $\pi_Y \circ F = F' \circ \pi_{X\times[0,1]}$, where $\pi_Y : Y \to Y/G$ and $\pi_{X\times[0,1]} : X \times [0,1] \to X/G \times [0,1]$ are the orbit maps.*

The first purpose of this paper is to prove the above theorem in the definable relative category.

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \cdots)$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$" and everything is considered in $\mathcal{M}$ unless otherwise stated. General references on o-minimal structures are [2], [3], see also [14]. In this paper, every definable map is assumed to be continuous. The semialgebraic category is the definable one of $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ and uncountably many definable categories exist [13]. Definable categories and definable C^r categories are studied in [4], [5], [6], [7], [8], [9], [10], [11] when G is compact or trivial.

A definable subset G of $\mathbb{R}^n$ is a *definable group* if G is a group and the group operations $G \times G \to G$ and $G \to G$ are definable. A *definable G set* means a pair consisting of a definable set X and a group action $\phi : G \times X \to X$ such that ϕ is definable. We say that a definable map between definable G sets is a *definable G map* if it is a G map.

In this paper, G denotes a compact definable group unless otherwise stated.

Theorem 1.2 (1.1 [11]). *Every definable G set has only finitely many orbit types.* □

For a definable G set X and a point $x \in X$, we can associate an orbit type (G/G_x) which is denoted by *type*(G/G_x). We say that $x, y \in X$ have the *same orbit type* if G_y is conjugate to G_x. We call the association $x \mapsto type(G/G_x)$ the *orbit structure* of X. The orbit structure of X induces an association $x \in X/G$ to *type*(G/G_x). This association is called the *orbit structure* of X/G.

Theorem 1.3. *Let G be a compact definable group, (X, X^-) a pair of definable G sets with X^- closed in X and Y a definable G set. Let $F' : X/G \times [0,1] \to Y/G$ be a definable homotopy which preserves an orbit structure. Suppose that $F'|X/G \times \{0\} \cup X^-/G \times [0,1]$ can be lifted to a definable G map $F_0 : X \times \{0\} \cup X^- \times [0,1] \to Y$ such that $\pi_Y \circ F_0 = F' \circ \pi_{X\times[0,1]}$, where $\pi_{X\times[0,1]} : X\times[0,1] \to X/G\times[0,1]$ and $\pi_Y : Y \to Y/G$ denote the orbit maps. Then there exists a definable G extension $F : X \times [0,1] \to Y$ of F_0 such that $\pi_Y \circ F = F' \circ \pi_{X\times[0,1]}$.*

Conjecture 1.4. (**Covering Mapping Cylinder Conjecture. (P98 [1])**). *Let G be a compact Lie group and W a compact G space. Suppose that W/G has the form of a mapping cylinder with orbit structure constant along generators of the cylinder less the base. Then W is G homeomorphic to a mapping cylinder of a G map inducing the given mapping cylinder structure on W/G.*

Our second purpose of this paper is to prove the following theorem which is the relative definable version of the above conjecture.

Theorem 1.5. *Let (B, A) be a pair of definable sets with A closed in B. Let W be a definable G set over $B \times [0,1]$ with the orbit map $\pi : W \to B \times [0,1]$ such that the orbit structure induced from that of W is constant on each $\{b\} \times [0,1)$ for $b \in B$. Let (X, X^-) be a pair of definable G sets definably G homeomorphic to $(\pi^{-1}(B\times\{0\}), \pi^{-1}(A \times \{0\})$ with the orbit map $\pi_X : X \to B$. Suppose that a definable G map $\phi : X \times \{0\} \cup X^- \times [0,1] \to \pi^{-1}(B\times\{0\} \cup A\times[0,1])$ commutes with the orbit maps. Then ϕ has a definable G extension $\overline{\phi} : X \times [0,1] \to W$ commutes with the orbit maps.*

Let X, Y be definable sets and f a definable map from X to Y. We say that f is *definably proper* if for any compact definable subset C of Y, $f^{-1}(C)$ is compact. The following theorem is the definable version of Conjecture 1.4.

Theorem 1.6. *Let G be a compact definable group and W a definable G set. Suppose that the orbit space W/G has the form of a definable mapping cylinder defined by a definably proper map with the orbit structure constant along generators of the cylinder less the base. Then W is definably G homeomorphic to a definable mapping cylinder of a definably proper G map which induces the given definable mapping cylinder structure on W/G.*

2 . Preliminaries and proof of Theorem 1.3

A *complex* in $\mathbb{R}^n$ is a finite collection K of simplexes in $\mathbb{R}^n$ such that for all $\sigma_1, \sigma_2 \in K$, either $\overline{\sigma_1} \cap \overline{\sigma_2} = \emptyset$ or $\overline{\sigma_1} \cap \overline{\sigma_2} = \overline{\tau}$ for some common face τ of σ_1 and σ_2, where $\overline{\sigma_1}$ (resp. $\overline{\sigma_1}$, $\overline{\tau}$) denotes the closure of σ_1 (resp. σ_2, τ). Notice that τ is not required to belong to K. Let $A \subset \mathbb{R}^m$ be a definable set. A *definable triangulation* in $\mathbb{R}^n$ of A is a pair (ψ, K) consisting of a complex K in $\mathbb{R}^n$ and a definable homeomorphism $\psi : A \to |K|$. The triangulation is said to be *compatible with a definable subset* $B \subset A$ if B is a union of some elements of $\psi^{-1}(K)$.

Theorem 2.1. (**Definable triangulation theorem (e.g. 8.2.9 [2])**). *Let $S \subset \mathbb{R}^m$ be a definable set and let $S_1, S_2, \ldots, S_k$ be definable subsets of S. Then S has a triangulation in $\mathbb{R}^m$ compatible with $S_1, \ldots, S_k$.*

Definable fiber bundles are introduced in [9].

Theorem 2.2 ([11]). *Let G be a compact definable group and X a definable G set.*

(1) There exists a definable slice at every point of X and X can be covered by finitely many definable tubes.

(2) If X has only one orbit type (G/H), then $(X, p, X/G, G/H, N(H)/H)$ is a definable fiber bundle, where $p : X \to X/G$ is the orbit map and $N(H)$ denotes the normalizer of H in G.

By a way similar to the proof of I.3.3 [1], we have the following.

Lemma 2.3. *Let G be a compact definable group and X, Y definable G sets. Let $C \subset X$ be any definable closed subset and $\phi : C \to Y$ a definable map such that whenever c and gc are both in C (for some $g \in G$), then $\phi(gc) = g\phi(c)$. Then ϕ can be extended uniquely to a definable G map ϕ' from GC into Y.* □

Let X be a definable set and Y a definable subset of X. A *definable retraction* $r : X \to Y$ means a definable map $r : X \to Y$ such that $r|Y = id_Y$. A *definable strong reformation retract* from X to Y is a definable map $R : X \times [0,1] \to X$ such that $R(x, 0) = x$ for all $x \in X$, $r(y,t) = y$ for all $y \in Y, t \in [0,1]$ and $R(X, \{1\}) = Y$. In this case we say that X is *definably strong deformation retractible* to Y.

A definable set Z is *definably contractible* if there exist a point $z_0 \in Z$ and a definable map $F : Z \times [0,1] \to Z$ such that $F(z, 0) = z$ and $F(z, 1) = z_0$ for all $z \in Z$.

Proposition 2.4 (3.3 [11]). *Let X be a definable set and A a closed definable subset of X. Suppose that A is a definable strong deformation retract of X. Then for any definable open neighborhood U of A in X, there exist a definable closed neighborhood N of A in U and a definable map $\rho : X \to U$ such that $\rho|N = id$ and $\rho(X - N) \subset U - N$.* □

Proposition 2.5. *Let (B, A) be a pair of definable sets with A closed in B. Let (X, X^-) be a pair of definable G sets having (B, A) as their definable orbit spaces with the orbit map $\pi : (X, X^-) \to (B, A)$. Suppose that B is definably strongly deformation retractible to A and each of the connected components of $B - A$ is definably contractible. Moreover assume that the induced orbit structure of $B - A$ is constant over its components. Then every definable G map $\mu_A : X^- \to G/H$ can be extended equivariantly and definably to $\mu : X \to G/H$.*

Proof. By Theorem 2.2, $\pi : X - X^- \to B - A$ is a definable fiber bundle. Since each connected component of $B - A$ is definably contractible, we can find a definable section $s : B - A \to X - X^-$. Without loss of generality, we may assume that $B - A$ is connected. Let $type(G/K)$ be the orbit type occurred on $X - X^-$. Since $(X - X^-)^K \to B - A$ is a definable fiber bundle, we may suppose that $K \subset H$ and $s(B - A) \subset (X - X^-)^K$.

Let $S := s(B - A)$, $cl(S)$ the closure of S in X, and $cl(S)_A := X^- \cap cl(S)$. We now construct a definable retraction $r : cl(S) \to cl(S)_A$. Let $\tilde{U}$ be a regular definable neighborhood of $cl(S)_A$ in $cl(S)$ and $U := B - \pi(cl(S) - \tilde{U})$. Since G is compact, the orbit map is closed. Then $\pi^{-1}(U) \cap cl(S) = \tilde{U}$ and U is a definable neighborhood of A, which follows from this fact. Since A is a definable strong deformation retract of B and by Proposition 2.4, there exist a smaller neighborhood N of A contained in U and a definable map $\rho : B \to U$ such that $\rho(x) = x$ for all $x \in N$ and $\rho(B - N) \subset U - N$. We can lift ρ to the map r' of $cl(S)$, precisely, r' is defined by

$$r'(x) = \begin{cases} s \circ \rho \circ \pi(x), & x \in cl(S) - cl(S)_A \\ x, & x \in cl(S)_A \end{cases}.$$

Then $r'(cl(S)) \subset \tilde{U}$. Since $\tilde{U}$ is a regular neighborhood of $cl(S)_A$, there exists a definable retraction $\tilde{U} \to cl(S)_A$. Composing r' with this retraction, we have a definable retraction $r : cl(S) \to cl(S)_A$.

Let $\mu' : cl(S) \cup X^- \to G/H$ be the definable map defined by $r \cup \mu_A$. Since $cl(S) \subset X^K$ if $K \subset H$ and by Lemma 2.3, we can extend μ' to a definable G map $\mu : X = G(cl(S) \cup X^-) \to G/H, \mu(gx) = g\mu'(x)$. □

Proposition 2.6. *Let X be a definable G set and X^- a closed G invariant definable subset of X. Suppose that H is a definable subgroup of G and $\mu^- : X^- \to G/H$ is a definable G map. Then μ^- is extensible definably and equivariantly to a G invariant definable open neighborhood of X^-.*

Proof. Let $(B, A) := (X/G, X^-/G)$ and $\mu_A := \mu^-$. Let π denote the orbit map $X \to X/G$. By Theorem 2.1, there exists a definable triangulation (ψ, K) compatible with the its orbit structure and A. Then from the construction of K, the orbit structure of each simplex of K is constant on its interior. We replace K by its barycentric subdivision. Let U be the union of all open simplices of K which meet with A and $U^{(k)}$ the union of A with the k-th skeleton of U for $0 \le k \le n = \dim U$. We now successively extend $\mu_0 = \mu_A$ to μ_n defined on $\pi^{-1}(U^{(n)})$. If $n = 0$, then $U^{(0)} = A$ and there is nothing to prove. Since $U^{(k)} - U^{(k-1)}$ is a union of k-dimensional open simplices of K, connected components of $U^{(k)} - U^{(k-1)}$ have constant orbit structures. Since $U^{(k)}$ is definable strong deformation retractible to $U^{(k-1)}$ and by Proposition 2.5, μ_{k-1} is extensible to μ_k. Thus μ_n is the required one. □

Lemma 2.7. *Let X, Y be definable G sets and Z a definable subset of Y^G. Suppose that $\psi : X/G \to Y/G$ is a definable map and $\phi : X - \pi_X^{-1}(\phi^{-1}(\pi_Y(Z))) \to Y$ is a definable G map such that $\pi_Y \circ \phi = \psi \circ (\pi_X | X - \pi_X^{-1}(\phi^{-1}(\pi_Y(Z))))$. Then ϕ can be uniquely extended to a definable G map covering ψ.*

Proof. For $x \in \pi_X^{-1}(\psi^{-1}(\pi_y(Z)))$, uniquely define $\phi(x)$ by $\pi_Y^{-1}(\psi(\pi_X(x)))$. We now check the continuity of ϕ. Let $y \in Z$ and $x \in X$ such that $\phi(x) = y$. Let V be an open neighborhood of y. Since G is compact, we can take a smaller invariant open neighborhood of y. Thus we may assume that V is invariant. Since $\pi_Y(V)$ is an open subset of the orbit space Y/G, $(\psi \circ \pi_X)^{-1}(\pi_Y(V))$ is that neighborhood of x which maps into V by ϕ. □

Proposition 2.8 (6.3.8 [2]). *Let A, B be disjoint definable closed subsets of a definable set X. Then there exists a definable map $f : X \to [0,1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$* □

Theorem 2.9. *Let B be a definable set and W a definable G set with the orbit space $W/G = B \times [0,1]$ such that the orbit structure is constant on each $\{b\} \times [0,1]$ for $b \in B$. Then there exist a definable G set X with $X/G \cong B$ and a definable G homeomorphism $\phi : W \to X \times [0,1]$ such that $\pi = (\pi_X \times id) \circ \phi$, where $\pi : W \to B \times [0,1]$ and $\pi_X : X \to B$ denote the orbit maps. Moreover X can be taken to be $\pi^{-1}(B \times \{0\})$ and $\phi|\pi^{-1}(B \times \{0\}) : X \to X \times [0,1]$ to be the inclusion $x \mapsto (x, 0)$.*

The above theorem is a definable version of Theorem II.7.1 [1] originally stated in the topological category.

Proof. The last statement follows from the previous ones.

We proceed by double induction on the dimension of G and the number of connected components of G. To do so, we need the assumption that G is compact. Assume that the theorem is true for the action of all proper subgroups of G. Let F be the homeomorphic image of $\pi^{-1}(B \times \{0\})^G$ in B by the composition $p \circ \pi$, where $p : B \times [0,1] \to B$ denotes the projection. Thus $W^G = \pi^{-1}(F \times [0,1])$. The proof consists of four parts.

Part 1: We will cover $B - F$ by a finite number of definable open sets $\{U\}$ and construct definable G maps $\{\phi_U : \pi^{-1}(U \times [0,1]) \to G/H\}$.

By Theorem 2.1, we can take a definable triangulation K of B compatible with the orbit structure of B. For a fixed vertex u of K, let U be its star neighborhood. Let H be a subgroup of G such that $type(u) = type(G/H)$. This gives a definable G map $\mu_u : \pi^{-1}(u \times \{0\}) \to G/H$. We now extend μ_u to a definable G map $\mu_U : \pi^{-1}(U \times$

$[0,1]) \to G/H$ by successive applications of Proposition 2.6.

Let $U^{(i)}$ be the i-th skeleton of U. Consider $\pi^{-1}(U^{(i)} \times [0,1])$ for $0 \le i \le n$, where $n = \dim U$. Since $U^{(0)} \times I - \{u\} \times \{0\} = \{u\} \times (0,1]$ is definably contractible and the orbit structure on it is constant, we can apply Proposition 2.6 to get a definable extension $\mu_0 : \pi^{-1}(U^{(0)} \times [0,1]) \to G/H$. Since each connected component of $U^{(k)} \times [0,1] - U^{(k-1)} \times [0,1]$ is the product of an open simplex of K with $[0,1]$ and it satisfies the condition of Proposition 2.6, we have a definable G map $\mu_k : \pi^{-1}(U^{(k)} \times [0,1]) \to G/H$ as an extension of $\mu_{k-1} : \pi^{-1}(U^{(k-1)} \times [0,1]) \to G/H$. Taking $\mu_U := \mu_n$, Part 1 is complete because $X - F$ is covered by a finite number of $U's$ corresponding to the vertices of K in $B - F$.

Part 2: We consider the slice $S = \mu_U^{-1}(\{H\})$ $(H \neq G)$, where the theorem holds for the H space S by the inductive hypothesis. From this, we construct a definable G homeomorphism $\phi_U : \pi^{-1}(U \times [0,1]) \to \pi_X^{-1}(U) \times [0,1]$ covering the identity map of $U \times [0,1]$, where $\pi_X : X \to B$ denotes the orbit map of $X \to B$.

Let $u \in B-F$ be a vertex of K and U the star neighborhood of u in K. Let H and $\mu_U : \pi^{-1}(U \times [0,1]) \to G/H$ be as in Part 1 and let $S := \mu_U(\{H\})$ and $T := S \cap \pi^{-1}(B \times \{0\})$. Then S and T are definable H sets with $S/H = U \times [0,1]$ and $T/H = U \times \{0\}$. Let $\pi_S^H : S \to U \times [0,1]$ be the orbit map. Since $H \neq G$, by applying the inductive hypothesis to S, we have a definable H homeomorphism $\phi_U^H : S \to T \times [0,1]$ commuting with the corresponding orbit maps $S \to S/H = U \times [0,1]$ and $T \times [0,1] \to T/H \cong U \times [0,1]$. By composing definable G homeomorphisms $\pi^{-1}(U \times [0,1]) = GS \cong G \times_H S \cong G \times_H (T \times [0,1]) \cong (G \times_H T) \times [0,1] \cong \pi_X^{-1}(U) \times [0,1]$, we get a definable G homeomorphism $\phi_U : \pi^{-1}(U \times [0,1]) \to \pi^{-1}(U) \times [0,1]$.

Part 3: We paste $\phi'_U s$ continuously to prove that the theorem holds for the action over $(B - F) \times [0,1]$.

Since $B - F$ can be covered by finitely many definable open sets over which the theorem holds, we have only to construct a definable G homeomorphism $\phi_{U\cup V} : \pi^{-1}((U \cup V) \times [0,1]) \to \pi_X^{-1}(U \cup V) \times [0,1])$ commuting with the orbit maps.

Let $\psi = \phi_U \circ \phi_V^{-1} : \pi_X(U \cap V) \to \pi_X^{-1}(U \cap V) \times [0,1]$. Since ϕ_U and ϕ_V are the maps covering the identities on $U \times [0,1]$ and $V \times [0,1]$, respectively, and the orbit structure on each $\{b\} \times [0,1]$ is constant for $b \in U \cap V$, $\phi_U(\pi^{-1}(U \times \{t\}) = \pi^{-1}(U) \times \{t\}$ and $\phi_V(\pi^{-1}(V \times \{t\}) = \pi^{-1}(V) \times \{t\}$. Thus so is ψ. Moreover we may assume that ϕ_U and ϕ_V are identities on 0-level. Since ψ is t-level preserving, we can define $\psi_1 : \pi_X^{-1}(U \cap V) \times [0,1] \to \pi_X^{-1}(U \cap V)$ implicitly by $\psi(x,t) = (\psi_1(x,t), t)$. Then $\psi_1(x,0) = x$ because ϕ_U and ϕ_V are identities on 0-level. Since $U - V$ and $V - U$ are disjoint definable closed subsets of $U \cup V$ and by Proposition 2.8, there exists a definable function $f : U \cup V \to [0,1]$ such that $f = 1$ on a definable open neighborhood of $U - V$ and $f = 0$ on a definable open neighborhood of $V - U$. Define $\psi' : \pi_X^{-1}(u \cap V) \times [0,1] \to \pi_X^{-1}(U \cap V) \times [0,1]$ by $\psi'(x,t) = (\psi_1(x, f(\pi_X(x))t, t)$. Then ψ' is a definable G homeomorphism covering the identity of $(U \cap V) \times [0,1]$. Moreover ψ' is the identity on $\pi^{-1}((U \cap V) \times \{0\})$. Consider the map $\psi' \circ \phi_V : \pi^{-1}((U \cap V) \times [0,1]) \to \pi_X^{-1}(U \cap V) \times [0,1]$. If $p_1 \circ \pi(w)$ lies in the definable neighborhood of $U - V$ where $f = 1$, then $\psi' \circ \phi_U(w) = \psi \circ \phi_V(w) = \phi_U(w)$. If $p_1 \circ \pi(w)$ lies in the definable neighborhood of $V - U$ where $f = 0$, then $\psi' \circ \phi_V(w) = id \circ \phi_V(w) = \phi(w)$. Thus $\phi_{U\cup V} : \pi^{-1}((U \cup V) \times [0,1]) \to \pi_X^{-1}(U \cup V) \times [0,1]$ defined by

$$\phi_{U\cup V}(w) = \begin{cases} \phi_U(w) & \pi(w) \in (U - V) \times [0,1] \\ \psi' \circ \phi_V(w) & \pi(w) \in (U \cap V) \times [0,1] \\ \phi_V(w) & \pi(w) \in (V - U) \times [0,1] \end{cases}$$

is a well-defined definable G homeomorphism.

Part 4: We finally prove the theorem for the given action.

Since $\pi^{-1}(F \times [0,1])$ is the set of fixed points of G on W, it maps definably homeomorphically onto $F \times [0,1]$ via π. Similarly $\pi_X(F) \cong F$. Thus $\phi_F : \pi^{-1}(F \times [0,1]) \to \pi_X^{-1}(F) \times [0,1]$ is uniquely determined and it covers the identity of $F \times [0,1]$. Then $\phi := \phi_{B-F} \cup \phi_F$ is continuous by Lemma

2.7. Therefore it is the required definable G homeomorphism. □

Theorem 2.10. *Let (X, X^-) be a pair of definable G sets such that X^- is closed in X and $X \times [0,1]$ a definable G set such that G acts on $[0,1]$ trivially. Suppose that $f : X^- \times [0,1] \to X^- \times [0,1]$ is a definable G homeomorphism such that it commutes with the orbit maps and $f(x,0) = (x,0)$ for all $x \in X^-$. Then f is extensible to a definable G homeomorphism $\phi : X \times [0,1] \to X \times [0,1]$ which commutes with the orbit maps and $\phi(x,0) = x$ for all $x \in X$.*

Proof. By Theorem 2.1, there exists a definable triangulation K of B compatible with the orbit structure of B and A. By replacing K with its barycentric subdivision, we may assume that the star neighborhood of a vertex of K meets with A if and only if the vertex belongs to A.

Let $F = \pi(X^G)$. Fix a 0-simplex $u \in B - F$ in K. Let U be the star neighborhood of u and $U_A = U \cap A$. Then U_A is the star neighborhood of u in A. Let $\phi_{U_A} : \pi^{-1}(U_A) \times [0,1] \to \pi^{-1}(U_A) \times [0,1]$ be $\phi_A | \pi^{-1}(U_A) \times [0,1]$. Then the problem is reduced to the construction of a definable G homeomorphism $\phi_U : \pi^{-1}(U) \times [0,1] \to \pi^{-1}(U) \times [0,1]$ extending ϕ_{U_A} with the required properties under the inductive hypothesis on G. Thus we now construct ϕ_U.

If $u \notin A$, then we take $\phi_U = id$ and there is nothing to prove. Assume $u \in A$. If $type(u) = (G/H)$, then H is a proper definable subgroup of G because $u \in B - F$. Since U_A is the star neighborhood of u in A and by Proposition 2.6, we have a definable G map $\nu_{U_A} : \pi^{-1}(U_A) \to G/H$. By Proposition 2.6, we can extend ν_{U_A} to a definable G map $\nu_U : \pi^{-1}(U) \to G/H$. This gives a definable H slice $T := \nu_U(\{H\})$ and a definable H invariant subset $T \times [0,1] \subset \pi^{-1}(U) \times [0,1]$. Let $T_A := T \cap \pi^{-1}(A)$ and $S_A := \phi_{U_A}^{-1}(T_A \times [0,1]) \subset \pi^{-1}(U_A) \times [0,1]$. Then ϕ_{U_A} maps S_A definably H homeomorphically onto $T_A \times [0,1]$. Since S_A is the inverse image of $\{H\}$ by a definable map $\nu_A \circ p \circ \phi_{U_A}$, S_A is a definable H slice in $\pi^{-1}(U_A)$, where $p : \pi^{-1}(U_A) \times [0,1] \to \pi^{-1}(U_A)$ denotes the projection. By Proposition 2.6, we have a definable H slice S in $\pi^{-1}(U)$ containing S_A. Namely $\nu_A \circ p \circ \phi_{U_A} : \pi^{-1}(U_A) \times [0,1] \to G/H$ is extensible to a definable G map $\pi^{-1}(U) \times [0,1] \to G/H$ and S is obtained by the inverse image of $\{H\}$ by the extended G map.

We have two pairs of definable H slices (S, S_A) and $(T \times [0,1], T_A \times [0,1])$ in $(\pi^{-1}(U) \times [0,1], \pi^{-1}(U_A) \times [0,1])$, and ϕ_{U_A} maps S_A definably H homeomorphically onto $T_A \times [0,1]$. Applying Theorem 2.9 to the H space S with the orbit space $S/H = U \times [0,1]$, we have a definable H homeomorphism $\Psi : S \to T \times [0,1]$ commuting with the orbit maps. Note that $T = (\pi|S)^{-1}(U)$ and $\Psi(S_A) = T_A \times [0,1]$. Thus $\phi'_{U_A} = \phi_{U_A} \circ \Psi^{-1}$ maps $T_A \times [0,1]$ onto itself. Applying the inductive hypothesis, we can extend ϕ'_{U_A} to a definable H homeomorphism $\phi' : T \times [0,1] \to T \times [0,1]$ commuting with the orbit maps. By composing with Ψ, we have a definable H homeomorphism $S \times T \times [0,1]$ extending $\phi_{U_A}|S$. Since S and $T \times [0,1]$ are H slices in $\pi^{-1}(U)$, we obtain a definable G homeomorphism $\phi_U : \pi^{-1}(U) \cong G \times_H S \to G \times_H (T \times [0,1]) \cong \pi^{-1}(U)$ commuting with the orbit maps. Moreover ϕ_U extends ϕ_{U_A} because ϕ_U coincides with ϕ_{U_A} on S_A and $\pi^{-1}(U_A) = GS_A$. □

Proof of Theorem 1.3. The pull back of $F'|X/G \times \{0\} \cup X^-/G \times I : X/G \times \{0\} \cup X^-/G \times [0,1] \to Y/G$ is a definable G set. By the universal property of pull backs, there exists a unique definable G map $\psi_0 : X \times \{0\} \cup X^- \times [0,1] \to (F'|X/G \times \{0\} \cup X^-/G \times [0,1])^*(Y)$ defined by $\psi_0(x,t) = ((\pi_X(x), t), F_0(x,t))$ such that $p_Y \circ \psi_0 = F_0$ and $p_{X/G \times \{0\} \cup X^-/G \times [0,1]} \circ \psi_0 = \pi_{X \times [0,1]}$, where $p_Y : (F'|X/G \times \{0\} \cup X^-/G \times [0,1])^*(Y) \to Y$ and $p_{X/G \times \{0\} \cup X^-/G \times [0,1]} : (F'|X/G \times \{0\} \cup X^-/G \times [0,1])^*(Y) \to X/G \times \{0\} \cup X^-/G \times [0,1]$ denote the projections. Since F' preserves orbit structures, so does ψ_0. Hence ψ_0 is a definable G homeomorphism.

Let $W := (F')^*Y$. Then W is a definable G set with orbit space $W/G = X/G \times [0,1]$. Hence we have $F'|X/G \times \{0\} \cup X^-/G \times [0,1])^*(Y) = \pi_W^{-1}(X/G \times \{0\} \cup X^-/G \times [0,1])$, where $\pi_W : W \to X/G \times [0,1]$ denotes the orbit map. Thus ψ_0 gives a definable G homeomorphism $X \times \{0\} \cup X^{\times}[0,1] \to \pi_W^{-1}(X/G \times \{0\} \cup X^-/G \times [0,1])$. By Theorem 2.10, there exists a definable G homeomorphic extension $\psi : X \times [0,1] \to W$ of ψ_0. Thus by the pull back diagram, we have a definable G map $F : X \times [0,1] \to Y$ such that $p_Y \circ \psi = F$. □

3. Proof of Theorem 1.5 and 1.6

Proposition 3.1. *Let $f : X \to Y$ be a definable G map between definable G sets which covers a definable map $f' : X/G \to Y/G$.*

(1) f is surjective if and only if f' is surjective.

(2) f is proper if and only if f' is proper.

(3) If f' preserves the orbit structure, then f is a definable G homeomorphism if and only if f' is a definable homeomorphism.

Proof. (1) follows trivially. (2) follows from I.3.1 [1]. Note that f is bijective if and only if f' is bijective. (3) follows from this fact and the definition of the topology of X/G and Y/G. □

Remark that we cannot directly generalize the proof of Theorem 2.10 because difficulty arises at the fixed point set when we apply the inductive hypothesis on G.

Proof of Theorem 1.5. For simplicity, we identify (X, X^-) with $(\pi^{-1}(B\times\{0\}), \pi^{-1}(A\times\{0\})$. There are two types of orbit structures, obtained from the association $B \to B \times \{t\} \to \{type(G/H)\}, (0 \le t \le 1)$ and $B \to B \times \{1\} \to \{type(G/H)\}$. By Theorem 2.1, there exists a definable triangulation K compatible with both orbit structures. Thus we may assume that for every open simplex $int(\delta)$ of K, the orbit structures on $int(\delta) \times [0,1)$ and $int(\delta) \times \{1\}$ are constant respectively. Moreover we can take K to be compatible with A. We replace K by its barycentric subdivision.

We proceed by induction on the dimension of B. If $B = \emptyset$, then the theorem holds trivially. Assume that the theorem is true for $(n-1)$-dimensional definable orbit spaces. Thus we may assume that the result holds for $(n-1)$-skeleton of K.

Let δ be an n-simplex of K closed in B. By the inductive hypothesis, there exists a definable G map $\phi : \pi_X^{-1}(\partial\delta) \times [0,1] \cup \pi_X^{-1}(\delta) \times \{0\} \to \pi_W(\partial\delta \times [0,1] \cup \delta \times \{0\})$ commuting with the orbit maps. We have only to construct a definable G extension $\overline{\phi} : \pi_X(\delta) \times [0,1] \to \pi_W^{-1}(\delta \times [0,1])$ of ϕ covering the identity of $\delta \times [0,1]$. Thus the problem is reduced from (B, A) to $(\delta, \partial\delta)$ and we may set $W = \pi^{-1}(\delta \times [0,1]), (B, A) = (\delta, \partial\delta)$ and $(X, X^-) = (\pi^{-1}(\delta \times \{0\}, \pi^{-1}(\partial\delta \times \{0\})$.

We now construct $\overline{\phi}$ by the double induction on $\dim G$ and the number of connected components of G. To do so we need the compactness of G.

If $G = \{e\}$, then $\overline{\phi}$ is uniquely determined because $W \cong B \times [0,1] \cong X \times [0,1]$. Assume that $\overline{\phi}$ exists for any proper definable subgroup of G.

Let $Z_1 := \pi(W^G) \cap B \times \{1\}$. By the assumption on the orbit structure, $W^G = \emptyset$ if $Z_1 = \emptyset$.

(*Case I*). Suppose that $Z_1 = \emptyset$.

(*Step* 1). *Claim* There exists a definable G map $\mu : \pi_W^{-1}(B \times \{1\}) \to G/H$ for some proper subgroup H of G.

By the choice of K, there exists a sequence $\emptyset \subset \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k = B \times \{1\}$ of faces of $B \times \{1\}$ such that for each i, $\sigma_i - \sigma_{i-1}$ has a constant orbit type. Notice that σ_{i-1} is definably strong deformation retract of σ_i and $\sigma_i - \sigma_{i-1}$ is definably contractible. Let $type(\sigma_1) = G/H$. Then by successive applications of Proposition 2.5, we have a definable G map $\mu : \pi_W^{-1}(B \times \{1\}) \to G/H$, which prove Step 1.

(*Step* 2). *Claim* There exists a definable G extension $\overline{\mu} : W \to G/H$ of μ .

We now construct a definable G map $\nu : \pi_W^{-1}(A \times [0,1] \cup B \times \{1\}) \to G/H$. The restriction $\phi|X^- \times [0,1] : X^- \times [0,1] \to \pi_W^{-1}(A \times [0,1]) \subset W$ is surjective by Proposition 3.1. The map $X^- \times [0,1] \to X^- \times \{1\}$ defined by $(x,t) \mapsto (x,1)$ reduces to a definable G map $\pi_W^{-1}(A\times[0,1]) \to \pi_W^{-1}(A\times\{1\})$, which is denoted by ν'. Since ν' is the identity on $\pi_W^{-1}(A\times\{1\})$, it is identically extensible to a definable map $\nu'' : \pi_W^{-1}(A\times[0,1] \cup B\times\{1\}) \to \pi_W^{-1}(B\times\{1\})$. $\nu := \mu\circ\nu'' : \pi_W^{-1}(A\times[0,1]\cup B\times\{1\}) \to G/H$. Applying Proposition 2.5 to the pair $(B\times[0,1], A\times[0,1]\cup B\times\{1\})$, we have a definable G extension $\overline{\mu} : W = \pi_W^{-1}(B\times[0,1]) \to G/H$ of ν. Thus Step 2 is proved.

(*Step* 3). We now construct a definable G extension $\overline{\phi} : X\times[0,1] \to W$ of ϕ. Let $S = \mu^{-1}(\{eH\})$ and $T = S\cap X = S\cap\pi_W^{-1}(B\times\{0\})$. Note that $\phi((T\cap X^-)\times[0,1]) \subset S$ and $\overline{\mu}\circ\phi(x,t) = \nu\circ\phi(x,t) = \mu\circ\nu''\circ\phi(x,t)$ if $x\in X^-$. Thus $\overline{\mu}\circ\phi(x,t)$ is independent of t whenever $x\in X^-$. Hence $\overline{\mu}\circ\phi(x,t) = \{eH\}$ if $x\in T\cap X^-$. Since H is a proper definable subgroup of G and by the inductive hypothesis, there exists a definable H extension $\phi' : T\times[0,1] \to S$ of $\phi|(T\cap X^-)\times[0,1]$ such that ϕ' satisfies the required properties for H. Let $\overline{\phi} := G(\phi') : X\times[0,1] = G(T)\times[0,1] \to G(S) = W$. Then $\overline{\phi}$ commutes with the orbit maps and $\overline{\phi}$ extends $\phi|X^-\times[0,1]$. Replacing $\overline{\phi}$ by $\overline{\phi}\circ\Psi$, if necessary, where $\Psi : X\times[0,1] \to X\times[0,1]$ is defined by $\Psi(x,t) = (\overline{\phi}^{-1}\circ\phi(x,0), t)$ and $X\times\{0\}$ is identified with X. Then $\overline{\phi}$ is the required one.

(*Case II*). Suppose that $Z_1 \neq \emptyset$.

(*Step* 1). Let δ be a definable simplex definably homeomorphic to B and Z the definably homeomorphic part of Z_1 in δ obtained from the definable homeomorphism $\delta\to B\to B\times\{1\}$. Let C be the complementary face of Z in δ, namely C is the simple generated by the vertices not included in Z. Note that δ is not necessarily compact. Let $L\subset B\times[0,1]$ be the definably homeomorphic image of the convex hull in $\delta\times[0,1]$ generated by $\delta\times\{0\}$ and $Z\times\{1\}$ and U the definably homeomorphic image of the convex hull in $\delta\times[0,1]$ generated by $C\times\{0\}\cup\delta\times\{1\}$.

Let $q_L : \delta\times[0,1] \to L$ be the quotient map sending $x\times[0,1]$ to $x\times\{0\}$ for $x\in C$ and $q_U : \delta\times[1,2] \to U$ the quotient map sending $y\times[1,2]$ to $y\times\{1\}$ for $y\in Z$. Then q_L and q_U define a quotient map $q : \delta\times[0,2] \to B\times[0,1] = L\cup U$. Let $\alpha = p_2\circ q$, where $q_2 : B\times[0,1] \to [0,1]$ denotes the projection. Then $\alpha(C\times[0,1]) = \{0\}$ and $\alpha(C\times[1,2]) = \{1\}$ and $q(x,t) = (x,\alpha(x,t))$ after identifying δ with B.

Let $W^* := q^*(W)$ be the pull back of W by q and π_W, namely $W^* = \{(x,t)\in W\times(\delta\times[0,2]) | \pi_W(x) = q(t)\}$ Then W^* is a definable G set with the orbit map $\pi_{W^*} : W^* \to \delta\times[0,2], \pi_{W^*}(x,t) = t$.

The map $\tilde{q} : X\times[0,2] \to X\times[0,1]$ defined by $\tilde{q}(x,t) = (x,\alpha(\pi_X(x),t))$ is a definable G map, where $\pi_X(x)$ denotes the orbit of x. Then $\tilde{q}$ covers q and by the universal property of pullbacks, there exists a definable G map from $X\times[0,2]$ to $q^*(X\times[0,1])$. This map is a definable G homeomorphism because Proposition 3.1 and the orbit structures are preserved. Thus $X\times[0,2]$ definably G homeomorphic to $q^*(X\times[0,1]$.

We now translate $\phi : X\times\{0\}\cup X^-\times[0,1] \to \pi^{-1}(B\times\{0\}\cup A\times[0,1])$ covering the identity of $A\times[0,1]\cup B\times\{0\}$ to a definable G map $\phi^* : X^-\times[0,2]\cup X\times\{0\} \to \pi_W^{-1}(\partial\delta\times[0,1]\cup\delta\times\{0\}) \subset W^*$ covering the identity of $\partial\delta\times[0,2]\cup\delta\times\{0\}$. Since $(\tilde{q}|X^-\times[0,2]\cup X\times\{0\})\circ\phi$ covers q and by W^* is the pull back of q of W, there exists a definable G map $X^-\times[0,2]\cup X\times\{0\} \to W^*$. Shrinking the range, we have ϕ^*. Let $W_L^* = \pi_W^{-1}(\delta\times[0,1])$ and $W_U^* = \pi_W^{-1}(\delta\times[1,2])$. We now construct definable G maps $\tilde{\phi}_L^* : X\times[0,1] \to E_L^*$ and $\tilde{\phi}_U^* : X\times[1,2] \to W_U^*$ such that they are well attached and it gives an extension $\tilde{\phi}^* : X\times[0,2] \to W^*$ of ϕ^*.

(*Step* 2). Note that the orbit types in W_L^* are constant along $\pi_{W^*}^{-1}(\{x\}\times[0,1])$ for $x\in\delta$ except for $\pi_{W^*}^{-1}(Z\times\{1\})$. Since $\pi_{W^*}^{-1}(Z\times\{1\}) \subset (W_L^*)^G$ and by Lemma 2.7, the construction of $\tilde{\phi}_L^*$ on $X\times[0,1] - \pi_{X\times[0,1]}(Z\times\{1\})$ implies that of a definable G map $\tilde{\phi}_L^* : X\times[0,1] \to W_L^*$. We apply Theorem 1.3 with the following setting: $X := X\times[0,1] - \pi_{X\times[0,1]}(Z\times\{1\}), Y := W_L^*, X^- := X^-\times[0,1]\cup X\times\{0\} - \pi_{X\times[0,1]}(Z\times\{1\})$. Note that $X/G = \delta\times[0,1] - Z\times\{1\}$ and $Y/G =$

$\delta \times [0,1]$. Let $F' : X/G \times [0,1] \to Y/G$ be the homotopy defined by $F'((y,s),t) = (y, st)$. Then $F'(\cdot,1)$ is the inclusion map of $\delta \times [0,1] - Z \times \{1\} \subset \delta \times [0,1]$ and $F'(\cdot,0)$ maps $X/G \times \{0\}$ to $\delta \times \{0\}$. Define a definable G map $F_0 : X^- \times [0,1] \cup X \times \{0\} \to Y, F_0((x,s),t) := \phi^*(x, st)$. Then $\pi_Y \circ F_0 = F' \circ (\pi_X \times id_{[0,1]}|X \times \{0\} \cup X^- \times [0,1])$, where $\pi_X : X \to X/G$ and $\pi_Y : Y \to Y/G$ denote the orbit maps. Since F' preserves orbit structures and by Theorem 1.3, there exists a definable G homotopy $F : X \times [0,1] \to Y$ extending F_0 and covering F'.

Let $F_1(\cdot) := F(\cdot, 1)$. Then F_1 is a definable G map from $X \times [0,1] - \pi^{-1}_{X\times[0,1]}(Z \times \{1\})$ to W^*_L covering the inclusion $\delta \times [0,1] - Z \times \{1\} \to \delta \times [0,1]$ and $F_1(x,t) = \phi^*(x, 1 \cdot t) = \phi^*(x,t)$ for all $(x,t) \in X^- \times [0,1] - \pi^{-1}_{X\times[0,1]}(Z \times \{1\})$. Thus F_1 defines $\tilde{\phi}^*_L$ on $X \times [0,1] - \pi^{-1}_{X\times[0,1]}(Z \times \{1\})$.

(*Step* 3). Let $\phi^*_U : X^- \times [1,2] \cup X \times \{1\} \to W^*_U$ be the map defined by attaching two maps $\phi^*|X^- \times [1,2]$ and $\tilde{\phi}^*_L|X \times \{1\}$. Since the orbit type of $\pi^{-1}_W(\{x\} \times [0,1))$ in W is constant and greater than or equal to that of $\pi_W(\{x\} \times \{1\})$ in W, $W^*_U - \pi^{-1}_W(Z \times [1,2]) = \pi^{-1}_{W^*}(Z \times [1,2])$ has no fixed points of G. We can apply to Case 1 to get a definable G map $\overline{\phi}^*_U : (X - \pi^{-1}_X(Z)) \times [1,2] \to W^*_U - \pi^{-1}_{W^*}(Z \times [1,2])$ extending ϕ^*_U except for $\pi^{-1}_X(Z)$ and covering the identity of $(\delta - Z) \times [1,2]$ because $(W^*_U - \pi^{-1}_{W^*}(Z \times [1,2]))/G = (\delta - Z) \times [1,2]$ and $\delta - Z$ is a simplex. By Lemma 2.7, we can extend $\overline{\phi}$ to a definable G map uniquely defined on $\pi_X(Z) \times [1,2]$.

(*Step* 4). Consider the composition $X \times [0,2] \to W^* = q^*(W) \to W$, where the second map is obtained from the pull-back diagram. Then the composition $\phi' : X \times [0,2] \to W$ is a definable G map covering $q : \delta \times [0,2] \to B \times [0,1]$. Moreover $X \times [0,2]$ is also the pullback of $X \times [0,1]$ and $X \times [0,2] \to X \times [0,1]$ denotes $\tilde{q}$. If the map $X \times [0,1] \to W$ is well defined as a set theoretical function, then the proof is complete because q and $\tilde{q}$ are proper so that the map is definable. Since $\tilde{q}$ is injective on $\pi^{-1}_{X\times[0,2]}(int(\delta) \times [0,2])$, we only to check the well-definedness on $\pi^{-1}_{X\times[0,2]}(\partial\delta \times [0,2])$. Already $\phi|X^- \times [0,1] : X^- \times [0,1] \to \pi^{-1}(A \times [0,1])$ defines a well-defined map on $X^- \times [0,1] = \tilde{q}(\pi^{-1}_{X\times[0,2]}(\partial\delta \times [0,2]))$. Hence the map is well defined. □

Proof of Theorem 1.6. Let $f' : B \to B_1$ be a definable map defining the mapping cylinder structure of W/G. Let $F' : B \times [0,1] \to W/G$ be the definable map induced from the structure of mapping cylinder which is definably proper and consider the pull-back $(F')^*W \to B \times [0,1]$ of W by F'. Let $X := \pi^{-1}(B \times \{0\})$, where $\pi : W \to W/G$ denotes the orbit map. Then the map $(F')^*W \to B \times [0,1]$ satisfies the condition that the orbit structure is constant along each $\{b\} \times [0,1)$ for $b \in B$. By setting $X^- = \emptyset$ in Theorem 1.5, there exists a definable G map $X \times [0,1] \to (F')^*W$. Composing this map with the map $(F')^*W \to W$, we have a definable G map $F : X \times [0,1] \to W$. Since F covers the proper map $F' : B \times [0,1] \to W/G$ and by Proposition 3.1, F is proper. Let $Y := \pi^{-1}_W(B_1)$ and $f : X \to Y$ the definable G map defined by $f(w) = F(w,1)$. Then f is proper because Y is closed in W. On the other hand, $X \times [0,1] \cup Y \to W$ factors through $X \times [0,1] \cup Y \to M(f) \to W$ because the involved maps are all proper. Note that $M(f) \to W$ is bijective and covers the identity of W/G. Thus by Proposition 3.1, it is a definable G homeomorphism. □

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