Definable G fibrations

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Abstract

Let G be a definable group, $\eta = (E, p, X)$ a definable G fibration and $f, h : Y \to X$ definable G maps between definable G spaces. If f and h are definably G homotopic, then the induced definable G fibrations $f^*(\eta)$ and $h^*(\eta)$ are definable G fiber homotopy equivalent.

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1 . Introduction.

Let $\eta = (E, p, X)$ be a semialgebraic vector bundle over a semialgebraic set X and $f, h : Y \to X$ semialgebraic maps. If f and h are semialgebraically homotopic, then the induced semialgebraic vector bundles $f^*(\eta)$ and $h^*(\eta)$ are semialgebraically isomorphic (12.7.7 [1]). The equivariant (resp. The equivariant Nash, The topological) version of this result is studied in [2] (resp. [7], [6]).

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \cdots)$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$". General references on o-minimal structures are [4], [5], see also [15]. It is known in [14] that there exist uncountably many o-minimal expansions of $\mathcal{R}$. Any definable category is a generalization of the semialgebraic category and the definable category on $\mathcal{R}$ coincides with the semialgebraic one.

The equivariant definable (resp. The equivariant topological) fiber bundle version of the above result is considered in [9] (resp. [13]), and an equivariant definable category is studied in [10], [12], [11], [9], [8].

In this paper, we are concerned with the equivariant definable fibration version of 12. 7.7 [1], and all definable maps are assumed to be continuous.

Let G be a definable group. Let $p : E \to X$ be a surjective definable G map between definable G spaces. We say that (E, p, X) is a *definable G fibration* if for any definable G space Y, definable G maps $f : Y \to E$ and $F : Y \times [0, 1] \to X$ with $(p \circ f)(x) = F(x, 0)$ for all $x \in Y$, there exists a definable G map $H : Y \times [0, 1] \to E$ such that $p \circ H = F$ and $H(x, 0) = f(x)$ for all $x \in Y$. Let $\eta = (E, p, X), \eta' = (E', p', X)$ be definable G fibrations with the same base space. A definable G map $f : E \to E'$ is called a *definable G fiber map* if $p = p' \circ f$. Two definable G fiber maps $f, h : E \to E'$ are *definable G fiber homotopy equivalent* if there exists a definable G homotopy H_t :

$E \times [0,1] \to E'$ such that $p = p' \circ H_t, H_0 = f$ and $H_1 = h$.

Two definable G fibrations $\eta = (E, p, X)$, $\eta' = (E', p', X)$ with the same base space are called *definably G fiber homotopy equivalent* if there exist two definable G fiber maps $\phi : E \to E', \psi : E' \to E$ such that $\phi \circ \psi$ is definable G fiber homotopy equivalent to $id_{E'}$ and $\psi \circ \phi$ is definable G fiber homotopy equivalent to id_E.

Let G be a definable group. Two definable G maps $f, h : X \to Y$ between definable G spaces are *definably G homotopic* if there exists a definable G map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ for all $x \in X$ and $F(x,1) = h(x)$ for all $x \in X$. By [8], if G is a compact definable group, then for any two definable maps between definable G sets, they are G homotopy equivalent if and only if they are definably G homotopy equivalent.

Two definable G spaces X and Y are *definably G homotopy equivalent* if there exists two definable G maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h$ is definably G homotopic to id_Y and $h \circ f$ is definably G homotopic to id_X.

Theorem 1.1. *Let G be a definable group and $\eta = (E, p, X)$ a definable G fibration. Suppose that $f, h : Y \to X$ are definable G maps between definable G spaces which are definably G homotopic. Then the induced definable G fibrations $f^*(\eta)$ and $h^*(\eta)$ are definably G fiber homotopy equivalent.*

Corollary 1.2. *Let $\eta = (E, p, X)$ a definable fibration. Suppose that $f, h : Y \to X$ are definable maps between definable spaces which are definably homotopic. Then the induced definable fibrations $f^*(\eta)$ and $h^*(\eta)$ are definably fiber homotopy equivalent.*

Let Z be a definable G subspace of a definable G space X and $f_1, f_2 : X \to Y$ definable G maps such that $f_1(x) = f_2(x)$ for all $x \in Z$. We say that they are *definably G homotopic relative to Z* if there exists a definable G map $H : X \times [0,1] \to Y$ such that $H(x,0) = f_1(x)$ for all $x \in X$, $H(x,1) = f_2(x)$ for all $x \in X$ and $H(x,t) = f_1(x) = f_2(x)$ for all $x \in Z, t \in [0,1]$.

Theorem 1.3. *Let $\eta = (E, p, X)$ be a definable G fibration, Y a definable G space and $h_0, h_1 : Y \times [0,1] \to X$ definable G maps which are definably G homotopic relative to $Y \times \{0,1\}$. Suppose that $\Psi_\epsilon : E_0 \to E_1$, $(\epsilon = 0,1)$ are the definable G fiber homotopies from $\psi_0^*\eta = (E_0, p_0, Y)$ to $\psi_1^*\eta = (E_1, p_1, Y)$ obtained from definable G homotopies ψ_t^ϵ as in Theorem 1.1. Then Ψ_0 and Ψ_1 are definable G fiber homotopy equivalent. Here $\psi_\epsilon(x) = h_0(x,\epsilon) = h_1(x,\epsilon)$ and $\psi_t^\epsilon(x) = h_\epsilon(x,t)$, $(\epsilon = 0,1)$. In particular, the definable G fiber homotopy in Theorem 1.1 is unique up to definable G fiber homotopy equivalence.*

A definable path l of a definable space X is a definable map $l : [0,1] \to X$. A definable space X is *definably path connected* if for any two points $x, y \in X$, there exists a definable path $l : [0,1] \to X$ such that $l(0) = x$ and $l(1) = y$.

The following two corollaries are immediate consequences of Theorem 1.1 and 1.3.

Corollary 1.4. *Let $\eta = (E, p, X)$ be a definable fibration and l a definable path of X. Then there exist a definable homotopy equivalence $h = h(l) : p^{-1}(l(0)) \to p^{-1}(l(1))$ and a definable homotopy $h_t = h_t(l) : p^{-1}(l(0)) \to E$ such that $h_0 = i_{l(0)}, h_1 = i_{l(1)}$ and $p \circ h_t = l(t)$ for all $t \in [0,1]$, where for any $x \in X$, $i_x : p^{-1}(x) \to E$ denotes the inclusion. In particular, if X is definably connected, then all fibers of E are definably homotopy equivalent.*

Remark that the equivariant version of Corollary 1.4 is not always true because the fiber over $x \in X$ of η is not necessarily G invariant.

A definable G space X is *definably G contractible* if X is definably G homotopy equivalent to a fixed point $a \in X$.

Corollary 1.5. *Every definable G fibration over a definably G contractible definable G space X is definably G fiber homotopy equivalent to $X \times F$, where F is the fiber over a.*

Theorem 1.6. *Every definable fiber bundle (E, p, X, F, K) admits the covering homotopy property for all compact Hausdorff definable spaces. Namely for any definable map f from a compact Hausdorff definable space Y to E and for any definable homotopy $\phi_t : Y \to X$ such that $p \circ f = \phi_0$, there exists a definable homotopy $H_t : Y \to E$ such that $p \circ H_t = \phi_t$ and $H_0 = f$.*

Theorem 1.6 shows that definable fibrations are some kind of generalizations of definable fiber bundles.

In the rest of Introduction, we restrict our attention to definable sets.

Let X, Y be two definable sets and $x_0 \in X$. Let $f, h : X \to Y$ be definable maps and let $u : [0,1] \to Y$ be a definable path. We say that f is *definably homotopic* to h *along* u if there exists a definable map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ for all $x \in X$, $F(x,1) = h(x)$ for all $x \in X$ and $F(x_0,t) = u(t)$, and write $f \sim_u h$.

Let Y be a definably path connected definable set. The definable fundamental group $\pi_1^{def}(Y, y_0)$ is defined by the definable homotopy classes $[(S^1, 0), (Y, y_0)]$.

Let X be a definable set and A a definable subset of X. A definable map $i : A \to X$ satisfies *the definable homotopy extension property* if for any definable set Y, for any definable map $f : A \times [0,1] \to Y$ and for any definable map $F : X \to Y$ such that $F \circ i(x) = f(x,0)$ for all $x \in A$, there exists a definable map $H : X \times [0,1] \to Y$ such that $H \circ (i \times id_{[0,1]}) = f$ and $H(x,0) = F(x)$ for all $x \in X$. A base point x_0 of a definable set X is *non-degenerate* if the inclusion $\{x_0\} \to X$ satisfies the definable homotopy extension property.

Let $(X, x_0), (Y, y_0)$ be based definable sets. Two based definable maps $f, h : (X, x_0) \to (Y, y_0)$ are *based definably homotopic* if there exists a definable map $H : (X, x_0) \times [0,1] \to (Y, y_0)$ such that $H(x,0) = f(x)$ for all $x \in X$, $H(x,1) = h(x)$ for all $x \in X$ and $H(x_0,t) = y_0$ for any $t \in [0,1]$. Using based definable homotopies, we can define the based definable homotopy classes $[X, Y]_0$.

By Lemma 4.1, $[u] \in \pi_1^{def}(Y, y_0)$ and $[f] \in [X, Y]_0$, define $[u][f]$ to be $[f_1]$, where f_1 is any definable map such that $f \sim_u f_1$.

Theorem 1.7. *Let X, Y be based definable sets with non-degenerate base points. Then $\pi_1^{def}(Y, y_0)$ acts on the based definable homotopy classes $[X, Y]_0$ and the definable homotopy classes $[X, Y]$, and if Y is definably path connected, then there exists a bijection between $[X, Y]_0/\pi_1^{def}(Y, y_0)$ and $[X, Y]$.*

A definably connected definable set X is *definably simply connected* if $\pi_1^{def}(X, x_0)$ is trivial for some $x_0 \in X$.

Corollary 1.8. *Let X, Y be based definably path connected definable sets with non-degenerate base points.*

(1) A based definable map $X \to Y$ is definably null-homotopic if and only if it is based definably null-homotopic.

(2) If Y is definably simply connected, then the forgetting map $[X, Y]_0 \to [X, Y]$ is bijective.

Notice that the forgetting map in Corollary 1.8 is not always injective or surjective in general.

2. Proof of Theorem 1.1 and 1.3.

A *definable space* is an object obtained by pasting finitely many definable sets together along definable open subsets, and definable maps between definable spaces are defined similarly (see Chapter 10 [4]). Definable spaces are generalizations of semialgebraic spaces in the sense of [3].

A group G is defined abstractly to be a *definable group* if G is a Hausdorff definable space and the group operations $G \times G \to G, G \to G$ are definable. By a fundamental result on topological groups, every T_1 topological group is a regular space. Thus by 10.1.8 [4], a definable group G can be definably imbeddable into some $\mathbb{R}^n$. Hence definable groups defined abstractly coincide with definable groups defined ordinary.

Definition 2.1. Let G be a definable group.

(1) A *definable G space* is a pair (X, θ) consisting of a definable space X and a group action $\theta : G \times X \to X$ which is definable. For simplicity of notation, we write X for (X, θ).

(2) Let X and Y be definable G spaces. A definable map $f : X \to Y$ is called a *definable G map* if it is a G map. We say that X and Y are *definably G homeomorphic* if there exist definable G maps $h : X \to Y$ and $k : Y \to X$ such that $h \circ k = id_Y$ and $k \circ h = id_X$.

The definition of induced definable G fibrations shows the following two lemmas.

Lemma 2.2. *Let $\eta = (E, p, X), \eta_1 = (E_1, p_1, X_1)$ be definable G fibrations and $F_1 : E \to E_1$ a definable G fiber map from η to η_1. Suppose that $\psi : X' \to X, \psi' : X_1 \to X'$ are definable G maps and the induced map of F_1 is $\psi_1 := \psi \circ \psi' : X_1 \to X$. Then there exists a unique definable G fiber map $F' : E_1 \to E'$ inducing ψ' such that $F_1 = F \circ F'$. Here F is the induced map of ψ.*

Lemma 2.3. *Let $\psi^*(\eta) = (E', p', X')$ be the induced definable G fibration from $\eta = (E, p, X)$ via $\psi : X' \to X$ and $F : E' \to E$ a definable fiber G map. Then two definable fiber G maps $F_0, F_1 : \eta_1 = (E_1, p_1, X_1) \to \psi^*(\eta)$ are definable fiber G homotopy equivalent if and only if $F \circ F_0, F \circ F_1$ are definable fiber G homotopy equivalent.*

Proof of Theorem 1.1. By assumption, there exists a definable G homotopy $\psi_t : Y \to X$ such that $\psi_0 = f$ and $\psi_1 = h$.

Consider definable G fibrations $\psi_\epsilon^* \eta = (E_\epsilon, p_\epsilon, Y)$, definable G fiber maps $\Psi_\epsilon : E_\epsilon \to E$ and $\epsilon = 0, 1$.

Applying the covering homotopy property to a definable G map $\Psi_0 : E_0 \to E$ and a definable G homotopy $\psi_t \circ p_0 : E_0 \to X$, there exists a definable G homotopy $\Phi_t^0 : E_0 \to E$ such that $\Phi_0^0 = \Psi_0, p \circ \Phi_t^0 = \psi_t \circ p_0$. Then a definable G fiber map $\Phi_1^0 : E_0 \to E$ induces ψ_1. By Lemma 2.2, there exists a unique definable G fiber map $\Phi : E_0 \to E_1$ such that $\Psi_1 \circ \Phi = \Phi_1^0$. Hence we now prove that Φ is a definable G fiber homotopy equivalence.

Applying the covering homotopy property to a definable G map $\Psi_1 : E_1 \to E$ and a definable G homotopy $\psi_t \circ p_1 : E_1 \to X$, there exist a definable G homotopy $\Phi_t^1 : E_1 \to E$ such that $\Phi_1^1 = \Psi_1, p \circ \Phi_t^1 = \psi_t \circ p_1$.

By the above argument, we have a definable G fiber map $\Phi' : E_1 \to E_0$ such that $\Psi_0 \circ \Phi' = \Phi_0^1$

Let $I = [0, 1], J = I \times \{1\} \cup \{0, 1\} \times I \subset I^2$. Let $f' : E_0 \times J \to E, g : E_0 \times I^2 \to B, f'(x, 0, t) = \Phi_t^0(x), f'(x, 1, t) = \Phi_t^1 \circ \Phi(x), f'(x, s, 1) = \Phi_1^0(x), g(x, s, t) = \psi_t \circ p_0(x)$. Then $p \circ f' = g|E_0 \times J$. On the other hand, $E_0 \times I^2$ is definably G homeomorphic to $E_0 \times J \times I$. By the covering homotopy property, there exists a definable G lift $f : E_0 \times I^2 \to E$ of g. Then $f|E_0 \times I \times \{0\}$ is a definable G fiber homotopy equivalence between Ψ_0 and $\Psi_0 \circ \Phi' \circ \Phi$. By Lemma 2.3, $\Phi' \circ \Phi$ is definably G fiber homotopy equivalent to id_{E_0}. By a similar way, $\Phi \circ \Phi'$ is definably G fiber homotopy equivalent to id_{E_1}. Therefore Φ is a definable G fiber homotopy equivalence. □

Proof of Theorem 1.3. By assumption, for $\epsilon = 0, 1$, there exist definable G maps $H_\epsilon : E_0 \times [0, 1] \to E$ such that $H_\epsilon(x, 0) = \Psi_0(x), \Psi_\epsilon(x, 1) = \Psi_1 \circ \Phi^\epsilon(x), p \circ H_\epsilon = h_\epsilon \circ (p_0 \times id)$. Let $J = [0, 1] \times \{0\} \cup \{0, 1\} \times [0, 1]$, $f' : E_0 \times J \to E, f'(x, \epsilon, t) = H_\epsilon(x, t), f'(x, s, 0) = \Psi_0(x), x \in E_0, \epsilon = 0, 1, s, t \in [0, 1]$ and $g : E_0 \times [0, 1]^2 \to X$ the composition of $p_0 \times id$ with the given definable G homotopy from h_0 to h_1. Then $g|E_0 \times J = p \circ f'$. Since $E_0 \times [0, 1]^2$ and $E_0 \times J \times [0, 1]$ are definably G homeomorphic, applying the covering homotopy property, we have a definable G lift $f : E_0 \times [0, 1]^2 \to E$ of g as an extension of f'. Thus $f|E_0 \times [0, 1] \times \{0\}$ is a definable G fiber homotopy between $\Psi_1 \circ \Phi^0$ and $\Psi_1 \circ \Phi^1$. Therefore by Lemma 2.3, Φ^0 and Φ^1 are definably G fiber homotopy equivalent. □

3 . Proof of Theorem 1.6.

Recall the definition of definable fiber bundles [11].

Definition 3.1. (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

(a) The total space E is a definable space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p : E \to X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of η such that each U_i is a definable open subset of X and $\{U_i\}_i$ is a finite open covering of X. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where π_i stands for the projection $U_i \times F \to F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations *definable*.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\overline{f} : E \to E'$ is said to be a *definable fiber bundle morphism* if the following two conditions are satisfied:

(a) There exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

(b) For any i, j such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in K, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

A definable fiber bundle morphism $\overline{f} : E \to E'$ is called a *definable fiber bundle isomorphism* if $X = X'$, $f = id_X$ and there exists a definable fiber bundle morphism $\overline{f'} : E' \to E$ such that $f' = id_X$, $\overline{f} \circ \overline{f'} = id$, and $\overline{f'} \circ \overline{f} = id$.

In this section, we prove the following stronger version of Theorem 1.6.

Theorem 3.2. *Let $r : B \to Z$ be a definable map between compact Hausdorff definable spaces, $\eta = (E, p, X, F, K)$ a definable fiber bundle, $\phi_t : Z \to X$ a definable homotopy and $\Psi : B \to E$ a definable map such that $p \circ \Psi = \phi_0 \circ r$. Then there exists a definable homotopy $\Psi_t : B \to E$ such that*

(1) $p \circ \Psi_t = \phi_r \circ r$.

(2) $\Psi_0 = \Psi$

(3) If $\Psi | r^{-1}(z) : r^{-1}(z) \to p^{-1}(\phi_0(z))$ is a definable homeomorphism for some $z \in Z$, then for any t, $\Psi_t | r^{-1}(z) : r^{-1}(z) \to p^{-1}(\phi_t(z))$ is a definable homeomorphism.

(4) If for any two t_1, t_2 contained in some closed subinterval I' of $[0,1]$, $\phi_{t_1}(z) = \phi_{t_2}(z)$ for any $z \in Z$, then for any $t_1, t_2 \in I'$, $\Psi_{t_1}(x) = \Psi_{t_2}(x)$ for any $x \in r^{-1}(z)$.

In Theorem 3.2, taking $B = Z$ and $r = id_Z$, we have Theorem 1.6.

Proposition 3.3. *Let Z be a compact Hausdorff definable space, X a definable space, $F : Z \times [0,1] \to X$ a definable map and $\mathcal{U}$ a finite definable open cover of X. Then there exist a finite number of definable maps $\tau_\lambda : Z \to [0,1]$ such that*

(1) $\tau_\lambda(z) \leq \tau_{\lambda+1}(z)$ for all $z \in Z$.

(2) Let $Z_\lambda = \{z \in Z | \tau_\lambda(z) < \tau_{\lambda+1}(z)\}, Y_\lambda = \cup_{z \in Z_\lambda}(\{z\} \times [\tau_\lambda(z), \tau_{\lambda+1}(z)]) \subset Z \times [0,1]$. Then there exists $U \in \mathcal{U}$ such that $F(\overline{Y_\lambda}) \subset U$, where $\overline{Y_\lambda}$ denotes the closure of Y_λ in $Z \times [0,1]$.

(3) Let $z \in Z$. Assume that $\{\lambda | z \in Z_\lambda\}$ consists of $\lambda_0, \ldots, \lambda_n$ with $\lambda_0 < \cdots < \lambda_n$. Then $0 = \tau_{\lambda_0}(z) < \tau_{\lambda_0+1}(z) = \tau_{\lambda_1}(z) < \tau_{\lambda_1+1}(z) = \cdots = \tau_{\lambda_n}(z) < \tau_{\lambda_n+1}(z) = 1$.

To prove Proposition 3.3, we prepare the following lemma. Lemma 3.4 is obtained from 6.3.7 [4] and 6.3.8 [4] and the proofs of them work in the definable space setting.

Lemma 3.4. *Let Z be a definable space and W, V two definable open subsets of Z with $\overline{W} \subset V$, where $\overline{W}$ denotes the closure of W in Z. Then there exists a definable function $\rho : Z \to [0,1]$ such that $\rho(\overline{W}) = 1$ and $\rho(Z - V) = 0$.* □

Proof of Proposition 3.3. Since Z is a compact Hausdorff space, Z is normal. Moreover since for any $z \in Z$, $\{z\} \times [0,1]$ is compact, there exist a finite definable open cover $\{V_\nu\}_{\nu \in I}$ of Z and a finite partition $0 = t_{(\nu,0)} < t_{(\nu,1)} < \cdots < t_{(\nu,n(\nu))} = 1$ such that

(*) $F(\overline{V_\nu} \times [t_{(\nu,i-1)}, t_{(\nu,i)}])$ is contained in some $U \in \mathcal{U}$.

By 6.3.6 [4], there exists a finite definable open cover $\{W_\nu\}$ of Z such that $\overline{W_\nu} \subset V_\nu$. By Lemma 3.3, we can find a definable function $\rho_\nu : Z \to [0,1]$ such that $\rho_\nu(\overline{W_\nu}) = 1$ and $\rho_\nu(Z - V_\nu) = 0$. Let $\sigma_{(\nu,i)} : Z \to [0,1]$ be $\sigma_{(\nu,i)}(z) = \min(\rho_\nu(z), t_{(\nu,i)})$. Then each $\sigma_{(\nu,i)}$ is definable and satisfies

(**) (1) $\sigma_{(\nu,i-1)}(z) \leq \sigma_{(\nu,i)}(z)$.

(2) $\sigma_{(\nu,i)}(Z - V_\nu) = 0$.

(3) $t_{(\nu,i)} < \rho_\nu(z)$ and $z \in V_\nu$ if $\sigma_{(\nu,i)}(z) < \sigma_{(\nu,i+1)}(z)$.

(4) $\sigma_{(\nu,0)}(z) = 0$.

(5) For any $z \in Z$, there exists ν such that $\sigma_{(\nu,n(\nu))}(z) = 1$.

Let Λ be a finite set $\{(\nu,i) | \nu \in I, 0 \leq i \leq n(\nu)\}$ with the lexicographic order. Then for any $\lambda \in \Lambda$, we define $\tau_\lambda(z) = \max_{\mu \leq \lambda} \sigma_\mu(z)$. Then each τ_λ is definable. We now prove that $\{\tau_\lambda\}_{\lambda \in \Lambda}$ is the required family. Conditions (1) and (3) follow from (*) and (**).

By the definition, $Z_{(\nu,n(\nu))} = \emptyset$ and $Z_{(\nu,i)} \subset V_\lambda$. Assume $Z_{(\nu,i)} \neq \emptyset$. Then $i < \nu$ and $\sigma_{(\nu,i)}(z) < \sigma_{(\nu,i+1)}(z)$ for any $z \in Z_{(\nu,i)}$. Hence $[\tau_{(\nu,i)}(z), \tau_{(\nu,i+1)}(z)] \subset [t_{(\nu,i)}, t_{(\nu,i+1)}]$.

On the other hand,

$$\begin{aligned}
\overline{Y_{(\nu,i)}} &= \overline{\cup_{z \in Z_{(\nu,i)}}(\{z\} \times [\tau_{(\nu,i)}(z), \tau_{(\nu,i+1)}(z)])} \\
&\subset \overline{\cup_{z \in Z_{(\nu,i)}}(\{z\} \times [t_{(\nu,i)}, t_{(\nu,i+1)}])} \\
&= \overline{Z_{(\nu,i)} \times [t_{(\nu,i)}, t_{(\nu,i+1)}]} \\
&= \overline{Z_{(\nu,i)}} \times [t_{(\nu,i)}, t_{(\nu,i+1)}] \subset \overline{V_\nu} \times [t_{(\nu,i)}, t_{(\nu,i+1)}].
\end{aligned}$$

Thus Condition (2) follows from (*). □

Proof of Theorem 3.2. Let $F : Z \times [0,1] \to X$ be $F(z,t) = \phi_t(z)$ and $\mathcal{U}$ a finite family of definable coordinate neighborhoods of η. By Proposition 3.3, there exists a finite family of definable functions $\tau_\lambda : Z \to [0,1]$. Take a definable coordinate neighborhood U_λ and its definable homeomorphism $\phi_\lambda : U_\lambda \times F \to p^{-1}(U_\lambda)$ of η such that $F(\overline{Y_\lambda}) \subset U_\lambda$. Let $q_\lambda : U_\lambda \times F \to F$ denotes the projection. Let $z \in Z$. Using the notation in Proposition 3.3, let $I_i = [\tau_{\lambda_i}(z), \tau_{\lambda_1+1}(z)]$, $0 \leq i \leq n$. We define a definable map $H_{z,i} : r^{-1}(z) \times I_i \to E$ to be

$$\begin{cases} H_{z,0}(x,t) = \phi_{\lambda_0}(F(z,t), q_{\lambda_0} \circ \phi_{\lambda_0}^{-1} \circ \Psi(x)), \\ H_{z,i}(x,t) = \phi_{\lambda_i}(F(z,t), q_\lambda \circ \phi_{\lambda_i}^{-1} \circ H_{z,i-1}(x, \\ \tau_{\lambda_i}(z))), i > 0. \end{cases}$$

Then $H_{z,i}(x, \tau_{\lambda_i}(z)) = H_{z,i-1}(x, \tau_{\lambda_i}(z))$, and thus a definable map $H_z : r^{-1}(z) \times [0,1] \to E$ is defined by $H_z(x,t) = H_{z,i}(x,t)$. Hence the map $H : B \times [0,1] \to E$ defined by $H(x,t) = H_{r(x)}(x,t)$ is definable and $\Psi_t(x) = H(x,t)$ satisfies our requirements. □

4. Proof of Theorem 1.7.

Lemma 4.1. *Let $(X, x_0), (Y, y_0)$ be based definable sets with non-degenerate base points.*

(1) Given a definable map $f_0 : X \to Y$ and a definable path u in Y starting at $f_0(x_0)$, $f_0 \sim_u f_1$ for some f_1.

(2) Suppose that $f_0 \sim_u f_1, f_0 \sim_v f_2$ and u is definably homotopic to v relative to $[0,1] \times \{0,1\}$. Then $f_1 \sim_{const} f_2$.

(3) $f_0 \sim_u f_1, f_1 \sim_v f_2$ implies $f_0 \sim_{uv} f_2$.

Proof of Theorem 1.7. We first verify that the action is well defined. By Lemma 4.1 (2), it is independent of the choice of representative of $[u]$. Suppose that $[f] = [g] \in [X, Y]_0$ and $g \sim_u g_1$. Then $f_1 \sim_{u^{-1}} f \sim_{const} g \sim_u g_1$. Thus by Lemma 4.1 (2) and (3), f_1 and g_1 are based definably homotopic. By Lemma 4.1 (3), this defines an action of $\pi_1^{def}(Y, y_0)$ on $[X, Y]_0$. Let $F : [X, Y]_0 \to [X, Y]$ be the forgetting map. Then $F([u][f]) = [f]$, and if $F([f_0]) = F([f_1])$, then there exists u such that $[u][f_0] = [f_1]$. Since Y is definably path connected and by Lemma 4.1 (3), F is surjective. $\square$

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