Definable C^r groups and proper definable actions

Tomohiro Kawakami
Department of Mathematics, Faculty of Education, Wakayama University,
Sakaedani Wakayama 640-8510, Japan
kawa@center.wakayama-u.ac.jp

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Abstract

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. Let G be a definable C^r group and H a definable C^r subgroup of G. We prove that if $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition or $0 \le r < \infty$, then the orbit map $\pi : G \to G/H$ has a principal definable C^r fiber bundle structure.

Moreover we prove that every proper definable G set X has only finitely many orbit types and that X can be covered by finitely many definable tubes.

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1 . Introduction.

Nash manifolds, Nash maps and Nash groups have been studied in [15], [17], [18], [11], [5].

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \cdots)$ denote an o-minimal expansion on the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$". General references on o-minimal structures are [2], [3], [16]. Any definable category is a generalization of the semialgebraic category and the definable category on $\mathcal{R}$ coincides with the semialgebraic one. It is known in [14] that there exist uncountably many o-minimal expansions on $\mathcal{R}$. Nash manifolds, Nash maps and Nash groups are definable C^ω manifolds, definable C^ω maps and definable C^ω groups in $\mathcal{R}$, respectively, and we can replace C^ω by C^∞. Everything is considered in $\mathcal{M}$ and all definable maps are assumed to be continuous.

In this paper we prove that if $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition or $0 \le r < \infty$, then the orbit map $\pi : G \to G/H$ has a principal definable C^r fiber bundle structure. Moreover we prove that every proper definable G set X has only finitely many orbit types and that X can be covered by finitely many definable tubes.

Theorem 1.1. *Let G be a definable C^r group, H a definable C^r subgroup of G and K a definable C^r subgroup of H. Let $\pi : G/K \to G/H$ be the map induced by the inclusion of cosets. If $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition or $0 \le r < \infty$, then $(G/K, \pi, G/H, H/K, H/K_0)$ is a definable C^r fiber bundle, where K_0 denotes the largest* *subgroup of K normal in H.*

Corollary 1.2. *Let G be a definable C^r group, H a definable C^r subgroup of G, $\pi : G \to G/H$ the orbit map. If $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition or $0 \leq r < \infty$, then $(G, \pi, G/H, H)$ is a principal definable C^r fiber bundle.*

The C^∞ (resp. C^0) version of this corollary is obtained in [7] (resp. [9]).

Let G be a definable group. A *definable G set* means a pair consisting of a definable set X and a group action $\phi : G \times X \to X$ such that ϕ is definable. A definable map between definable sets is called *definably proper* if the inverse image of every compact definable set is compact. We call a definable G set X a *proper definable G set* if the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (gx, x)$ is definably proper.

Let G be a definable group. We can define *orbit types* as well as G is compact.

Theorem 1.3. *Let G be a definable group. Then every proper definable G set has only finitely many orbit types.*

Let G be a definable group, X a proper definable G set and H a compact definable subgroup of G. A subset S of X is called a *definable H slice* if GS is a definable open subset of X and there exists a definable G map $f : GS \to G/H$ such that $f^{-1}(eH) = S$. We call GS a *definable tube*. For each $x \in X$, a *definable slice at x* means a definable G_x slice S in X such that $x \in S$.

Theorem 1.4. *Let G be a definable group and X a proper definable G set. Then there exists a definable slice at every point and X can be covered by finitely many definable tubes.*

A special case of Theorem 1.4 is proved in [9].

Finiteness of definable tubes in Theorem 1.4 and the proof of 1.2 [9] prove the following corollary.

Corollary 1.5. *Let G be a definable group and X a proper definable G set. If X has only one orbit type G/H, then $(X, \pi, X/G, G/H, N(H)/H)$ is a definable fiber bundle, where $\pi : X \to X/G$ is the orbit map and $N(H)$ denotes the normalizer of H in G.*

A definable subgroup of some $GL_n(\mathbb{R})$ is called a *definable linear group*. By [12] and [13], we have the following theorem.

Theorem 1.6. *Let G be a definable linear group and X a proper definable G set. Then X is definably G imbeddable into some representation of G.*

2. Preliminaries.

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets. We say that a continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m$) is definable. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be definable open sets and $0 \leq r \leq \omega$. A C^r map $f : U \to V$ is called a *definable C^r map* if it is definable. A definable C^r map $h : U \to V$ is called a *definable C^r diffeomorphism* (a *definable homeomorphism* if $r = 0$) if there exists a definable C^r map $k : V \to U$ such that $h \circ k = id_V$ and $k \circ h = id_U$.

Recall definable C^r manifolds, definable C^r groups and definable $C^r G$ manifolds [10].

Definition 2.1. Suppose that $0 \leq r \leq \omega$.

(1) A definable subset X of $\mathbb{R}^n$ is called a *d-dimensional definable C^r submanifold of* $\mathbb{R}^n$ if for any $x \in X$ there exists a definable C^r diffeomorphism (a definable homeomorphism if $r = 0$) ϕ_x from some definable open neighborhood U_x of the origin in $\mathbb{R}^n$ onto some definable open neighborhood V_x of x in $\mathbb{R}^n$ such that $\phi_x(0) = x, \phi(\mathbb{R}^d \cap U_x) = X \cap V_x$. Here $\mathbb{R}^d$ denotes the subset of $\mathbb{R}^n$ those whose the last $(n-d)$ components are zero.

(2) A *definable C^r manifold X of dimension d* is a C^r manifold with a finite system of charts $\{\phi_i : U_i \to \mathbb{R}^d\}$ such that for each i and j, $\phi_i(U_i \cap U_j)$ is a definable open subset of $\mathbb{R}^d$ and the map $\phi_j \circ \phi_i^{-1}|\phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable

C^r diffeomorphism (a definable homeomorphism if $r = 0$). We call this atlas *definable* C^r. Definable C^r manifolds with compatible atlases are identified. A subset Y of X is said to be *definable* if each $\phi_i(U_i \cap Y)$ is a definable subset of $\mathbb{R}^d$. A definable subset Z of X is called a *k-dimensional definable* C^r *submanifold* of X if each point $x \in Z$ there exist a definable open neighborhood U_x of x in X and a definable C^r diffeomorphism ϕ_x from U_x to some definable open subset V_x of $\mathbb{R}^d$ such that $\phi_x(x) = 0$ and $U_x \cap Y = \phi_x^{-1}(\mathbb{R}^k \cap V_x)$, where $\mathbb{R}^k \subset \mathbb{R}^d$ is the vectors whose last $(d-k)$ components are zero.

(3) Let X (resp. Y) be a definable C^r manifold with definable C^r charts $\{\phi_i : U_i \to \mathbb{R}^n\}_i$ (resp. $\{\psi_j : V_j \to \mathbb{R}^m\}_j$). A C^r map $f : X \to Y$ is said to be a *definable* C^r *map* if for any i and j $\phi_i(f^{-1}(V_j) \cap U_i)$ is definable and open in $\mathbb{R}^n$ and the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \to \mathbb{R}^m$ is a definable C^r map.

(4) Let X and Y be definable C^r manifolds. We say that X is *definably* C^r *diffeomorphic to* Y (*definably homeomorphic* to Y if $r = 0$) if one can find definable C^r maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h = id_Y$ and $h \circ f = id_X$.

(5) A definable C^r manifold is said to be *affine* if it is definably C^r diffeomorphic (definably homeomorphic if $r = 0$) to a definable C^r submanifold of some $\mathbb{R}^l$.

Remark 2.2. (a) The definition of definable subsets of a definable C^r manifold X does not depend on the choice of definable C^r charts of X.

(b) By o-minimality, a definable C^r submanifold of $\mathbb{R}^n$ admits a finite family of definable C^r charts, thus it is of course a definable C^r manifold. In Definition 2.1 (2), by o-minimality, Z is covered by finitely many such neighborhoods. Hence Z is also a definable C^r manifold.

(c) We can consider a definable C^r manifold X with possibly different dimensions on different connected components of X. In this paper, we assume that every connected component of a definable C^r manifold has the same dimension.

Definition 2.3. Let $0 \le r \le \omega$.

(1) A group G is called a *definable* C^r *group* (resp. an *affine definable* C^r *group*) if G is a definable C^r manifold (resp. an affine definable C^r manifold) and that the multiplication $G \times G \to G$ and the inversion $G \to G$ are definable C^r maps.

Let G be a definable C^r group.

(2) A subgroup H of G is called *definable* if it is a definable subset of G.

(3) A subgroup K of G is said to be a *definable* C^r *subgroup* of G if K is a definable C^r submanifold of G.

(4) A group homomorphism (resp. An group isomorphism) between two definable C^r groups is a *definable* C^r *group homomorphism* (resp. a *definable* C^r *group isomorphism*) if it is a definable C^r map (resp. a definable C^r diffeomorphism (a definable homeomorphism if $r = 0$)).

(5) A *definable* C^rG *manifold* is a pair (X, θ) consisting of a definable C^r manifold X and a group action θ of G on X such that $\theta : G \times X \to X$ is a definable C^r map. For simplicity of notation, we write X instead of (X, θ).

(6) A definable C^r diffeomorphism (resp. A definable homeomorphism) is a *definable* C^rG *diffeomorphism* (resp. a *definable* G *homeomorphism*) if it is a G map.

Remark that every definable subgroup of a definable group is closed. The converse is not true because $\mathbb{Z}$ is a closed subgroup of $\mathbb{R}$ but not definable.

Example 2.4. *Affine algebraic groups and the identity component of an affine algebraic group are definable* C^ω *groups. Moreover every Nash group is a definable* C^ω *group.*

3 . Definable C^r groups and definable C^r fiber bundles.

It is known that $\mathcal{M}$ admits the C^r cell decomposition for any non-negative integer

(7.3.3.2 [2]). We say that $\mathcal{M}$ admits *the* C^ω (resp. C^∞) *cell decomposition* if we can take $r=\omega$ (resp. $r=\infty$).

Theorem 3.1 (2.15 [7]). *Let G be a definable C^r group. If $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition or $0 \le r < \infty$, then every definable subgroup of G is a definable C^r subgroup of G.*

Proposition 3.2. *Let $f: G \to H$ be a definable C^r group homomorphism. If $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition or $0 \le r < \infty$, then*

(1) $Ker f$ is a normal definable C^r subgroup of G.

(2) $f(G)$ is a definable C^r subgroup of H.

(3) If H_1 is a definable C^r subgroup of H, then $f^{-1}(H_1)$ is a definable C^r subgroup of G.

Recall the definition of definable fiber bundles [9].

Definition 3.3. (1) A topological fiber bundle $\eta=(E,p,X,F,K)$ is called a *definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

(a) The total space E is a definable space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p: E \to X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of η such that each U_i is a definable open subset of X and $\{U_i\}_i$ is a finite open covering of X. For any $x \in U_i$, let $\phi_{i,x}: p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where π_i stands for the projection $U_i \times F \to F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations *definable*.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta=(E,p,X,F,K)$ and $\zeta=(E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i,\phi_i\}_i$ and $\{V_j,\psi_j\}_j$, respectively. A definable map $\overline{f}: E \to E'$ is said to be a *definable morphism* if the following two conditions are satisfied:

(a) The map $\overline{f}$ covers a definable map, namely there exists a definable map $f: X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

(b) For any i,j and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in K, and $f_{ij}: U_i \cap f^{-1}(V_j) \to K$ is a definable map.

We say that a bijective definable morphism $\overline{f}: E \to E'$ is a *definable equivalence* if it covers a definable homeomorphism $f: X \to X'$ and $(\overline{f})^{-1}: E' \to E$ is a definable morphism covering $f^{-1}: X' \to X$. A definable equivalence $\overline{f}: E \to E'$ is called a *definable isomorphism* if $X = X'$ and $f = id_X$.

(3) A continuous section $s: X \to E$ of a definable fiber bundle $\eta=(E,p,X,F,K)$ is a *definable section* if for any i, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

(4) We say that a definable fiber bundle $\eta=(E,p,X,F,K)$ is a *principal definable fiber bundle* if $F=K$ and the K action on F is defined by the multiplication of K. We write (E,p,X,K) for (E,p,X,F,K).

Recall the definition of definable C^r fiber bundles [7].

Definition 3.4 ([7]). Let $1 \le r \le \omega$.

(1) A definable fiber bundle $\eta=(E,p,X,F,K)$ is a *definable C^r fiber bundle* if the total space E and the base space X

are definable C^r manifolds, the structure group K is a definable C^r group, the fiber F is a definable C^rK manifold with an effective action, the projection p is a definable C^r map and all transition functions of η are definable C^r maps. A *principal definable C^r fiber bundle* is defined similarly.

(2) *Definable C^r morphisms*, *definable C^r equivalences*, *definable C^r isomorphisms* between definable C^r fiber bundles and *definable C^r sections* of a definable C^r fiber bundle are defined similarly.

Proof of Theorem 1.1. By 1.3 [7], G/K and G/H are definable C^r manifolds and the projections $p_1 : G \to G/K$ and $p_2 : G \to G/H$ are definable C^r maps with $p_2 = \pi \circ p_1$. By the construction of K_0, $K_0 = \cap_{h\in H} hKh^{-1}$. Thus K_0 is a normal definable subgroup of K. Hence K_0 is a normal definable C^r subgroup of H, H/K_0 is a definable C^r group by 1.3 [7] and it acts effectively on H/K. Moreover the map $\phi : H/K_0 \times H/K \to H/K$ defined by $\phi(hK_0, h'K) = hh'K$ gives an action of H/K_0 on H/K. This map is definable, and it is also of class C^r because p_1 and p_2 are piecewise definably C^r trivial (1.1 [7]) and thus ϕ gives a definable C^r action of H/K_0 on H/K.

By the proof of 1.3 [7], there exist a definable open subset U of G/H and $g_1, \dots, g_n \in G$ such that $\{U_{g_i}\}_{i=1}^n$ is the definable coordinate neighborhoods of G/H, where $U_{g_i} = g_iU$. Let $f : U \to G$ be a definable C^r section and let $f_{g_i} : U_{g_i} \to G, f_{g_i}(x) = g_if(g_i^{-1}x)$. Then $p_2 \circ f_{g_i}(x) = x$. We can define a definable coordinate function $\phi_{g_i} : U_{g_i} \times H/K \to \pi^{-1}(U_{g_i}), \phi_{g_i}(x,y) = f_{g_i}(x)y$. This map is a definable C^r one and $\pi \circ \phi_{g_i}(x,y) = x$. The map $p_{g_i} : \pi^{-1}(U_{g_i}) \to H/K$ defined by $p_{g_i}(z) = (f_{g_i}(\pi(z)))^{-1}z$ is a definable C^r map such that $p_{g_i}\phi_{g_i}(x,y) = y, \phi_{g_i}(\pi(z), p_{g_i}(z)) = z$. Hence $\psi_{g_i} : \pi^{-1}(U_{g_i}) \to U_{g_i} \times H/K, \psi_{g_i}(z) = (\pi(z), p_{g_i}(z))$ is the inverse map of ϕ_{g_i} and ϕ_{g_i} is a definable C^r diffeomorphism between $U_{g_i} \times H/K$ and $\pi^{-1}(U_{g_i})$.

If $x \in U_{g_i} \cap U_{g_j}$, then $p_{g_j} \circ \phi_{g_i}(x,y) = (f_{g_j}(x))^{-1}(f_{g_i}(x)y) = ((f_{g_j}(x))^{-1}f_{g_i}(x))y$ is a left translation of y by $h'_{ij}(x) := (f_{g_j}(x))^{-1} f_{g_i}(x)$. Since $p_2 \circ f_{g_i}(x) = p_2 \circ f_{g_j}(x) = x$, $h'_{ij}(x) \in H$ and $h'_{ij} : U_{g_i} \cap U_{g_j} \to H$ is a definable C^r map. The coordinate transformation h_{ij} in $U_{g_i} \cap U_{g_j}$ is given by the composition of the projection $H \to H/K_0$ and h'_{ij}. □

Theorem 3.5. *Let G be a definable C^r group and X a definable C^rG manifold with a transitive action. If $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition or $0 \le r < \infty$, and $x \in X$, then G_x is a definable C^r subgroup of G and $f : G/G_x \to X, f(gG_x) = gx$ gives a definable C^rG diffeomorphism (a definable G homeomorphism if $r = 0$).*

Proof. Since $G_x = \{g \in G | gx = x\}$, G_x is a definable subgroup of G. Thus G_x is a definable C^r subgroup of G by Theorem 3.1. By fundamental facts on transformation groups, f is a bijective G map. Let $F : G \to X, F(g) = gx$. Then F is definable and the graph of f is the image of that of F by $\pi \times id_X$, where $\pi : G \to G_x$ denotes the orbit map. Thus f is definable. By the C^r cell decomposition, f is of class C^r in a definable open neighborhood of eG_x. Since the action is transitive, f is of class C^r. By the same argument, f^{-1} is of class C^r. □

Corollary 3.6. *Let $f : G \to H$ be a surjective definable C^r group homomorphism between definable C^r groups. If $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition or $0 \le r < \infty$, then the map $f : G/Kerf \to H$ defined by $f(gKerf) = f(g)$ is a definable C^r group isomorphism.* □

Corollary 3.7. *If $\mathcal{M}$ admits the C^ω (resp. C^∞) cell decomposition or $0 \le r < \infty$, then every bijective definable C^r group homomorphism between definable C^r groups is a definable C^r group isomorphism.* □

Theorem 3.8. *Let $r = \omega$ or ∞ and G, H compact affine definable C^r groups. If $f : G \to H$ is a Lie group homomorphism, then f is a definable C^r group homomorphism.*

Proof. The graph $\Gamma(f)$ of f is a closed subgroup of a compact affine definable C^r

group $G \times H$. Thus $\Gamma(f)$ is a Lie subgroup of $G \times H$. Since $\Gamma(f)$ is compact, $\Gamma(f)$ admits an algebraic group structure compatible with the Lie group structure. Thus $\Gamma(f)$ is a definable C^r subgroup of $G \times H$ because $G \times H$ is affine. Therefore f is a definable C^r group homomorphism. □

Corollary 3.9. *Let $r = \omega$ or ∞ and G, H compact affine definable C^r groups.*

(1) If $f : G \to H$ is a Lie group isomorphism, then f is a definable C^r group isomorphism.

(2) G and H are topologically group isomorphic if and only if they are definable C^r group isomorphic.

Remark that in Theorem 3.8 and Corollary 3.9, we cannot drop the condition that G, H are affine.

Theorem 3.10. *Let $r = \omega$ or ∞ and H a definable C^r subgroup of a compact affine definable C^r group G. If $\mathcal{M}$ admits the C^r cell decomposition, then G/H is an affine definable C^rG manifold.*

Proof. By 1.3 [7], G/H is a definable C^rG manifold. Since H is closed in G, H is a compact affine definable C^r group. Since G is affine, we can assume that G is a definable subset of some $\mathbb{R}^n$. By 10.2.8 [2], G/H exists as a definable subset of some $\mathbb{R}^l$. Therefore G/H is affine. □

The Nash version of Theorem 3.10 is proved in [11].

By [8], if $0 \leq r < \infty$, then every definable C^r manifold is affine. Thus we have the following proposition.

Proposition 3.11. *If $0 \leq r < \infty$, then every definable C^r group is affine.* □

4. Proper definable actions.

We now define orbit types. We say that two homogeneous proper definable G sets are *equivalent* if they are definably G homeomorphic. Let (G/H) denote the equivalence class of G/H. The set of all equivalence classes of homogeneous proper definable G sets has a natural order defined as $(X) \geq (Y)$ if there exists a definable G map $X \to Y$. If $(X) = (G/H)$ and $(Y) = (G/K)$, then $(X) \geq (Y)$ if and only if H is conjugate to a definable subgroup of K. The reflexivity and the transitivity clearly hold and the anti-symmetry is true by the following lemma.

Lemma 4.1. *Let G be a definable group, H a definable subgroup of G and $g \in G$. If $gHg^{-1} \subset H$, then $gHg^{-1} = H$.*

Proof. Let H_0 denote the identity component of H. Then H_0 is a normal definable subgroup of H. Since H has only finitely many connected components, H/H_0 is a finite group. Since $gHg^{-1} \subset H$, $\phi_g : H/H_0 \to H/H_0, \phi_g(hH_0) = ghg^{-1}H_0$ is a well-defined map. Moreover ϕ_g is injective because $\psi_g|H$ is injective and $\psi_g(H_0) = H_0$, where $\psi_g : G \to G, \psi_g(x) = gxg^{-1}$. Thus ϕ_g is an automorphism. In particular $g(H/H_0)g^{-1} = H/H_0$. Therefore $gHg^{-1} = H$. □

Proof of Theorem 1.3. First assume that G is a definable solvable group. Since G acts on X properly, the isotropy subgroup G_x are definably compact. By [4], the uniform definable family $\{G_x | x \in X\}$ of definably compact definable subgroups of G is contained in a maximal definably compact definable subgroup K of G. By [4], this family is finite.

Second assume that G is definably simple. By [12] and [13], G is a linear semialgebraic group. Moreover we may assume that the G action $G \times X \to X$ is semialgebraic after replacing X by a suitable G/H such that $(X) = (G/H)$. In this setup the result follows from by transfer from the topological setting.

By [12] and by the definably simple case, the result holds for the definably semi-simple case. The general case follows from the solvable case and the semi-simple case. □

Let X be a definable set and A a definable subset of X. A *definable strong* *deformation retract from* X *to* A is a definable map $R : X \times [0,1] \to X$ such that $R(x,0) = x$ for all $x \in X$, $R(y,t) = y$ for all $y \in Y, t \in [0,1]$ and $R(X,1) = Y$.

Proposition 4.2. *Let* X *be a definable set and* A *a closed definable subset of* X. *Suppose that* A *is a definable strong deformation retract of* X. *Then for any definable open neighborhood* U *of* A *in* X, *there exist a definable closed neighborhood* N *of* A *in* U *and a definable map* $\rho : X \to U$ *such that* $\rho|N = id$ *and* $\rho(X - N) \subset U - N$.

To prove Proposition 4.2, we need the following lemma and theorem.

Lemma 4.3. *Let* X *be a definable set and* $f : X \to \mathbb{R}$ *(resp.* $g : X \to \mathbb{R}$*) a lower (resp. upper) semi-continuous function such that they have definable graphs and* $g(x) \leq f(x)$ *for all* $x \in X$. *Then there exists a definable function* $h : X \to \mathbb{R}$ *such that* $g(x) \leq h(x) \leq f(x)$ *for all* $x \in X$ *and* $g(x) < h(x) < f(x)$ *whenever* $g(x) < f(x)$.

Theorem 4.4. *Let* X *be a definable set and* A *a definable closed subset of* X. *Then every definable function* f *on* A *is extensible to a definable function* F *on* X.

The semialgebraic version of Theorem 4.4 is Theorem 3 [1].

Proof. By 3.4 [6], there exist a definable open neighborhood U of A in X and a definable map $r : U \to A$ such that $r|A = id$. Thus $f \circ r$ is a definable extension of f. Take a definable open neighborhood V of A in X with $\overline{V} \subset U$, $\overline{V}$ denotes the closure of V in X. By 6.3.7 [2], we can find a definable partition of unity $h_1, h_2 : X \to \mathbb{R}$ subordinate to $U, X - \overline{V}$. Then the definable function $F : X \to \mathbb{R}$ defined by $F(x) = h_1(x)(f \circ r(x))$ is the required function. □

Proof of Lemma 4.3. By the piecewise triviality theorem (9.1.7 [2]) and the definable triangulation theorem (8.2.9 [2]), there exists a definable triangulation (K, τ) of X such that f and g are continuous on the interior of each simplex of K. We identify K with X. We now construct $h : X \to \mathbb{R}$ by induction on the skeleta of K. Let $K^{(n)}$ denote the union of all simplexes of K whose dimension do not exceed n. Since K^0 is finitely many points, clearly we have the required function h. Assume that we have a definable function $h_1 : K^{(n-1)} \to \mathbb{R}$ such that $g(x) \leq h_1(x) \leq g(x)$ for all $x \in |K^{(n-1)}|$ and $g(x) < h(x) < f(x)$ if $g(x) \neq f(x)$.

Let δ be an n-dimensional simplex of K which is closed in K. Note that δ is not always compact. Then the proof is reduced to find $h : \delta \to \mathbb{R}$ satisfying the inequality condition and $h|\partial\delta = h_1$, where $\partial\delta$ means the boundary of δ.

We now first construct a lower semi-continuous function $f' : \delta \to \mathbb{R}$ and a upper semi-continuous function $g' : \delta \to \mathbb{R}$ such that they have definable graphs, $g(x) \leq g'(x) \leq f'(x) \leq f(x)$ for all δ and all the inequalities are strict if $g(x) \neq f(x)$. Let $\alpha : int(\delta) \to (0, \infty)$ be the distance between x and $\partial\delta$. Define $\alpha' : int(\delta) \to [0, \infty)$ by $\alpha'(x) = \min(\alpha(x), (f - g)(x)/3)$. Then α' is continuous and vanishes when it approaches the boundary of δ. Hence it is continuously extensible to δ. Define $f', g' : \delta \to \mathbb{R}$ by $f'(x) = f(x) - \alpha'(x)$ and $g'(x) = g(x) + \alpha'(x)$.

We now construct $h : \delta \to \mathbb{R}$ such that $g'(x) \leq h(x) \leq f'(x)$. By Theorem 4.4, h_1 has a definable extension $h_2 : \delta \to \mathbb{R}$. Since h_2 is an extension of h_1, $g'(x) \leq h_2(x) \leq f'(x)$ for all $x \in \partial\delta$. We now modify h_2 such that it satisfies the inequality on all δ. Let $\tilde{f} = \min(f', h_2)$ and $\tilde{g} = \max(g', h_2)$. We claim that $\tilde{f}$ and $\tilde{g}$ are continuous. Since f' and h_2 are continuous in the interior of δ, so is $\tilde{f}|int(\delta)$. Thus we have to show that $\tilde{f}|\partial\delta$ is continuous. Let $x \in \partial\delta$. For a given a, b with $a < \tilde{f}(x) < b$, we now prove that $\{y \in \delta | a < \tilde{f}(y) < b\}$ contains a neighborhood of x. Note that $\{y \in \delta | \min(h_2, f')(y) > a\} = \{y \in \delta | h_2(y) > a\} \cap \{y \in \delta | f'(y) > a\}$, $\{y \in \delta | \min(h_2, f')(y) < b\} = \{y \in \delta | h_2(y) < b\} \cup \{y \in \delta | f'(y) < b\}$. Then $\{y \in \delta | \min(h_2, f')(y) > a\}$ is open because h_2 and f' is lower semi-continuous, and $\{y \in \delta | \min(h_2, f')(y) < b\}$ contains a neighborhood of x since an open set $\{y \in \delta | h_2(y) < b\}$ contains

x. Since $\{y \in \delta | a < \tilde{f}(y) < b\} = \{y \in \delta | \min(h_2, f')(y) > a\} \cap \{y \in \delta | \min(h_2, f')(y) < b\}$, $\tilde{f}$ is continuous at x. Thus $\tilde{f}$ is continuous. Similarly $\tilde{g}$ is continuous.

Clearly $\tilde{f} \leq f'$. Let $h = \max(g', \tilde{f})$. Then by a way similar to the above, h is continuous, and h satisfies $g \leq g' \leq h \leq f' \leq f$. By the definition of f' and g', if $f(x) < g(x)$, then $f(x) < h(x) < g(x)$. □

Proof of Proposition 4.2. Let $R : X \times [0,1] \to X$ be a definable strong deformation retract from X to A. Let $g : X \to [0,1]$ be the function defined by $g(x) = \inf\{r \in [0,1] | F(x,t) \in U$ for all $t \in (r,1]\}$. Then g has the definable graph. We now prove that g is upper semi-continuous. We need to show that for every $a \in \mathbb{R}$, $\{x \in X | g(x) < a\}$ is open. For x_0 with $g(x_0) < a$, take b such that $g(x_0) < b < a$. By the definition of g, $R(x_0, t) \in U$ for all $t \in [b,1]$. Since $[b,1]$ is compact, there exists a definable open neighborhood V of x_0 such that $R(V \times [b,1]) \subset U$. Since $g(y) \leq b < a$, $g^{-1}(\{y < a\})$ is open. Hence g is upper semi-continuous.

Since $R(A \times [0,1]) = A \subset U$ and $[0,1]$ is compact, there exists a definable closed neighborhood N of A such that $R(N \times [0,1]) \subset U$. Let $f : X \to [0,1]$ be the function defined by $f(x) = \inf\{r \in [g(x),1] | R(x,r) \in N\}$. Then f is well defined, it has the definable graph, $g(x) = f(x) = 0$ for all $x \in N$ and $g(x) < f(x)$ for all $x \notin N$.

We now prove that f is lower semi-continuous. Let $x_0 \notin N$ and take a with $g(x_0) < a < f(x_0)$. Choose $b, c \in [0,1]$ such that $g(x_0) < b < a < c < f(x_0)$. Since g is upper semi-continuous, there exists a open neighborhood V of x_0 such that $g(x) < b$ whenever $x \in V$. Since N is closed and $[b,c]$ is compact, there exists a neighborhood V' of x_0 such that $R(V' \times [b,c]) \cap N = \emptyset$. This implies that if $x \in V'$ then $f(x) > a$. Hence f is lower semi-continuous on $X - N$. Since $f|N = 0$, f is lower semi-continuous on X.

By Lemma 4.3, there exists definable function h such that $g(x) \leq h(x) \leq g(x)$ for all $x \in X$ and the inequalities become strict whenever $g(x) \neq f(x)$. Let $\rho(x) = R(x, h(x))$. Then $\rho(x) = R(x,0) = 0$ for all N and if $x \notin N$ then $\rho(x) = R(x, h(x)) \in U - N$ because $g(x) < h(x) < f(x)$. □

Proof of Theorem 1.4. Let $\pi : X \to X/G$ be the orbit map. By Theorem 1.3, X has only finitely many orbit types $\{G/H_i | 1 \leq i \leq n\}$. Then the set $X(H_i)$ of all points in X whose orbit type is (G/H_i) is a definable G subset of X. Hence each $\pi(X(H_i))$ is a definable subset of X/G. By the piecewise triviality theorem (9.1.7 [2]), there exists a finite partition $\{B_j\}_{j=1}^m$ of X/G and for each i there exists a definable homeomorphism $k_j : \pi^{-1}(B_i) \to B_i \times \pi^{-1}(b_j)$ such that $f|f^{-1}(B_j) = p_j \circ k_j$, where $b_j \in B_j$ and p_j denotes the projection $B_j \times \pi^{-1}(b_j) \to B_j$. Applying the definable triangulation theorem (8.2.9 [2]), we have a definable triangulation (K, τ) of X/G compatible with $B_1, \dots, B_m$, $\pi(X(H_1)), \dots, \pi(X(H_n))$. We identify K with X/G. Then the following two properties hold.

(1) For each simplex $\delta \in K$, there exists a definable section $s : int(\delta) \to X$ of $\pi : X \to X/G$.

(2) Every point in $s(int(\delta))$ has the same isotropy subgroup.

Let $x_0 \in X$ and $b_0 = \pi(x_0)$. Let $St(b_0)$ denote the open star neighborhood of b_0 in $X/G = K$ and let $St^{(k)}(b_0)$ be the k-th skeleton of $St(b_0)$. We now construct a definable G map $\psi : \pi^{-1}(St(b_0)) \to G/H$, where $H = G_{x_0}$. We proceed by induction on k. If $k = 0$, then $\pi^{-1}(St^{(0)}(b_0)) = G(x_0)$. Thus there exists a canonical definable G homeomorphism $\psi^{(0)} : G(x_0) \to G/H$. Assume that we have a definable G map $\psi^{(k-1)} : \pi^{-1}(St^{(k-1)}(b_0)) \to G/H$. It is enough to consider a k-dimensional simplex δ such that $int(\delta) \subset St^{(k)}(b_0)$. For notational convenience, we simply write δ instead of $\delta \cap St^{(k)}(b_0)$. Let $A = \delta \cap St^{(k-1)}(b_0)$. By (1) there exists a definable section $s : int(\delta) \to X$ of π. Let $\tilde{\delta}$ denote the closure of $s(int(\delta))$ in $\pi^{-1}(\delta)$.

We now prove that there exists a definable retraction $\tilde{r} : \tilde{\delta} \to \tilde{\delta} \cap \pi^{-1}(A)$.

First triangulate $\tilde{\delta}$ and take the open regular neighborhood $\tilde{U}$ of $\tilde{\delta} \cap \pi^{-1}(A)$ in $\tilde{\delta}$. Since

$\tilde{U}$ is open, $U := \pi(\tilde{U})$ is open in δ. Since A is a definable strong deformation retract of δ, Applying Proposition 4.2, there exist a closed definable neighborhood N of A in U and a definable map $\rho : \delta \to U$ such that $\rho(x) = x$ for all $x \in N$ and $\rho(\delta - N) \subset U - N$. The map $r' : \tilde{\delta} \to \pi^{-1}(U) \cap \tilde{\delta} = \tilde{U}$ defined by

$$r'(x) = \begin{cases} s \circ \rho \circ \pi(x), & x \in \tilde{\delta} - \pi^{-1}(A) \\ x, & x \in \tilde{\delta} \cap \pi^{-1}(A) \end{cases}$$

is definable because $\rho|N = id$ and $s \circ \rho \circ \pi(x) = x$ for all $x \in \pi^{-1}(N - A) \cap \tilde{\delta}$. Since the regular neighborhood $\tilde{U}$ has a definable retraction to $\tilde{\delta} \cap \pi^{-1}(A)$, composing this retraction, we have a definable retraction $\tilde{r} : \tilde{\delta} \to \tilde{\delta} \cap \pi^{-1}(A)$. Since any element in $\pi^{-1}(\delta)$ is of the form gx for some $g \in G$ and $x \in \tilde{\delta}$, we can extend $r := \psi^{(k-1)} \circ \tilde{r} : \tilde{\delta} \to G/H$ to a map $r_G : \pi^{-1}(\delta) \to G/H, gx \mapsto gr(x)$. Then r_G is a well-defined G map with definable graph. Since r_G is a G map, r_G is continuous. Hence r_G is a definable G map and this completes the inductive construction of a definable G map $\psi : \pi^{-1}(St(b_0)) \to G/H$. This shows the existence of a definable tube and a definable slice at x_0. Since the triangulation K is finite, X can be covered by finitely many definable tubes. □

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