Relative EHP-sequence

Kaoru MORISUGI
Wakayama University

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Abstract

This paper is expository and a record of two talks which I gave at Seoul in Korea, 2007 (Cf. [10]) and at Halifax in Canada, 2008.

Let B be the mapping cone of a map $\varphi : A \to X$. Then, Toda(1956) defined a map $T : \Omega(B, X) \to \Omega^2(\Sigma A \wedge B)$ where $\Omega(B, X)$ is the homotopy fiber of the inclusion map $X \to B$. Boardman-Steer(1967) and Ganea(1968), independently, constructed a map $\mu : \Omega B \to \Omega^2(\Sigma A \wedge B)$ using the coaction map of B. Boardman-Steer obtained a very useful result about the functional cup products. These maps, in fact, factor through $\Omega^2(\Sigma A \times B, \Sigma A \vee B)$, so they are "delicate Hopf invariants". I describe these construction and show that they are essentially the same and that these invariants have some useful properties related to the following James's old homotopy exact sequence(1954) for some range.

$$\pi_i(B, X) \xrightarrow{p'_*} \pi_i(\Sigma A) \xrightarrow{H_\varphi} \pi_i(\Sigma(X \wedge A)) \xrightarrow{\Delta} \pi_{i-1}(B, X) \to \cdots ,$$

here the above exact sequence can be regarded as "relative EHP-sequence".

1 Motivation

Assume that X is $(m-1)$ connected. Let $X \cup_\varphi e^n$ be the mapping cone of a map $\varphi : S^{n-1} \to X$. James[7](1954) proved the following sequence is exact for $k \leq 2m-3$ and $k \leq 2n-5$.

$$\pi_{n+k+1}(S^n) \xrightarrow{H_\varphi} \pi_{k+1}(X) \xrightarrow{[\gamma_n,]} \pi_{n+k}(X \cup_\varphi e^n, X) \xrightarrow{p'_*} \pi_{n+k}(S^n) \xrightarrow{H_\varphi} \cdots$$

Here H_φ is the following composite

$\pi_*(S^n) \xrightarrow{h_2} \pi_*(S^{n-1} \wedge S^n) \xrightarrow{(\varphi\wedge 1)_*} \pi_*(X \wedge S^n) \xleftarrow[\cong]{E^n} \pi_{*-n}(X)$, where h_2 is the James Hopf invariant. The element $\gamma_n \in \pi_n(X \cup_\varphi e^n, X)$ is the characteristic map of the cell e^n and $[\gamma_n,]$ means Relative Whitehead product.

We call the above sequence as *James exact sequence*.

It is easy to see that if $k \leq n-3$ and $k \leq 2m-3$, then

$$\pi_{n+k}(X \cup_\varphi e^n, X) \cong \pi_{k+1}(X) \oplus \pi_{n+k}(S^n). \qquad (1)$$

Specially let $k = m-1$(assume that $m \leq n-2$). Let $B = X \cup_\varphi e^n$. Take an element $\beta \in \pi_{n+m-1}(B)$ and consider the natural homomorphism

$j_* : \pi_{n+m-1}(B) \to \pi_{n+m-1}(B, X) = \pi_m(X) \oplus \pi_{n+m-1}(S^n)$.

Suppose that $\pi_m(X) \cong \mathbb{Z}$ with a generator $i_m : S^m \to X$, then James(1957)[9] or K.Yamaguchi(2004)[14] showed that under some conditions, the cup product in $H^*(B \cup_\beta e^{n+m})$ is related to James exact sequence as follows:

The following equation holds:

$$j_*(\beta) = x[\gamma_n, i_m] + y \qquad \text{for some integer } x \in \mathbb{Z},$$

where y is the component of $\pi_{n+k}(S^n)$ in the decomposition (1) of $j_*(\beta)$
if and only if

$$u_m u_n = x u_{n+m} \text{ in } H^*(B \cup_\beta e^{n+m}),$$

where it is assumed that $H^k(B \cup e^{n+m}) \cong \mathbb{Z}$ for $k = m$, n or $n+m$ with generator u_k corresponding to the bottom cell S^m, e^n and e^{n+m}, respectively.

My naive questions are:

Q1. Why and how is the relative Whitehead product related to the functional cup products?

Q2. What is the James exact sequence?

The purpose of this paper is to answer these questions and extend the James (and Yamaguchi)'s result to the more general situations.

In order to state the results, I need some notation and preliminaries.

2 Preliminaries

We are working on the based homotopy category. In this section we recall various Hopf invariants.

1. For a pair of spaces (X, A), we denote the homotopy fiber of the inclusion $A \to X$ by $\Omega(X, A)$.

$$\Omega(X, A) = \{(a, \omega) \in A \times X^I \mid \omega(0) = a \text{ and } \omega(1) = *\},$$

so there exists a fiber sequence:

$$\cdots \to \Omega X \xrightarrow{j} \Omega(X, A) \xrightarrow{\partial} A \to X.$$

The loop space $\Omega(\Omega(X, A))$ is sometimes abbreviated as $\Omega^2(X, A)$.

There exists a natural map $p' : \Omega(X, A) \to \Omega(X/A)$ defined by $p'(\omega)(t) = [\omega(t)]$, which satisfies the following commutative diagram:

$$\begin{tikzcd}
\Omega X \arrow[r, "j"] \arrow[d, equal] & \Omega(X, A) \arrow[r, "\partial"] \arrow[d, "p'"] & A \arrow[r, "i"] \arrow[d, "-E"] & X \\
\Omega X \arrow[r, "\Omega p"] & \Omega(X/A) \arrow[r, "\Omega\partial"] & \Omega\Sigma A &
\end{tikzcd}$$

where $E : A \to \Omega\Sigma A$ is a suspension map, the upper line is a fiber sequence and $A \xrightarrow{i} X \xrightarrow{p} X/A \xrightarrow{\partial} \Sigma A$ is a cofiber sequence.

2. The relative homotopy group $\pi_{k+1}(X, A)$ is canonically identified to the $\pi_k(\Omega(X, A))$. Moreover any base point preserving map $f : Y \to \Omega(X, A)$ can be regarded as a base point preserving map of pairs $\tilde{f} : (CY, Y) \to (X, A)$, where CY stands for the reduced cone of Y

which is defined from $Y \times I$ by collapsing $Y \times 1 \cup y_0 \times I$. Specially there exists a canonical map $e : A \to \Omega(X \cup CA, X)$ which satisfies the following commutative diagram:

$$\begin{tikzcd} & X \\ A \arrow[ur, "i"] \arrow[r, dotted, "\exists e"] \arrow[dr, "E"'] & \Omega(X \cup CA, X) \arrow[u, "\partial"'] \arrow[d, "p'"] \\ & \Omega\Sigma A \end{tikzcd}$$

3. For spaces X and Y, the homotopy fiber $\Omega(X \times Y, X \vee Y)$ is sometimes denoted by $X\flat Y$. Ganea showed that $X\flat Y = \Omega X * \Omega Y$(the unreduced join) which is homotopy equivalent to $\Sigma\Omega X \wedge \Omega Y$

4. There exists a map $R : \Omega(X \vee Y) \to \Omega(X\flat Y)$ such that

$$R \circ \Omega\partial = id_{\Omega(X\flat Y)},$$

$$\Omega(i_2 \circ p_2) + \Omega\partial \circ R + \Omega(i_1 \circ p_1) = id_{\Omega(X \vee Y)},$$

where i_j is the inclusion maps into the j-th factor of the wedge and and p_j is the projection map from the wedge to the j-th factor, respectively.

$$\begin{tikzcd} \Omega(X\flat Y) \arrow[r, "\Omega\partial"] & \Omega(X \vee Y) \arrow[l, bend right, "\exists R"'] \arrow[r, "\Omega j"] & \Omega(X \times Y) \end{tikzcd}$$

The above choice of R is convenient for later use with regard to Toda's relative Hopf invariant(modified) and for looking for the relation between "Hopf invariants" and reduced diagonal maps.

5. For a cofiber sequence $A \to X \to B = X \cup CA$, let $\nabla : B \to \Sigma A \vee B$ be the co-action map of the mapping cone $B = X \cup CA$. Ganea[4] calls the following composite

$$\mathbb{H} : \Omega B \xrightarrow{\Omega\nabla} \Omega(\Sigma A \vee B) \xrightarrow{R} \Omega((\Sigma A)\flat B) = \Omega(\Omega(\Sigma A \times B, \Sigma A \vee B))$$

as "delicate Hopf invariant" and

$$\mathbb{H}' : \Omega B \xrightarrow{\Omega\nabla} \Omega(\Sigma A \vee B) \xrightarrow{R} \Omega^2(\Sigma A \times B, \Sigma A \vee B) \xrightarrow{\Omega p'} \Omega^2(\Sigma A \wedge B)$$

as "crude Hopf invariant".

Remark 2.1. $\Omega j : \Omega(X \vee Y) \to \Omega(X \times Y)$ has a right inverse $s = \Omega(i_1 \circ p_1) + \Omega(i_2 \circ p_2)$. Here i_j and p_j are the inclusions and the projections into or from the product space. Ganea's original choice of R is characterized by

$$\Omega\partial \circ R + s \circ \Omega j = id_{\Omega(X \vee Y)}$$

$$\begin{tikzcd} \Omega(X\flat Y) \arrow[r, "\Omega\partial"] & \Omega(X \vee Y) \arrow[l, bend right, "\exists R"'] \arrow[r, "\Omega j"] & \Omega(X \times Y) \arrow[l, bend left, "\exists s"] \end{tikzcd}$$

6. Let $B = X \cup_\varphi CA$. About 1956, H. Toda[13] constructed the "relative Hopf invariant" as the following composite:

$$h_T : \Omega(B, X) \xrightarrow{\exists Q'} \Omega((\Sigma A)\flat B) \xrightarrow{\Omega p'} \Omega^2(\Sigma A \wedge B).$$

More precisely, as Ganea observed, the above map Q' can be described as follows. Let $h : \Omega(\Sigma A \vee B, B) \to \Omega(\Sigma A \vee B)$ be defined by $h(\omega) = -i_2 \circ p_2 \circ \omega + \omega$, by using path-addition. Then we have the following commutative diagram up to homotopy:

$$\begin{tikzcd}
\Omega B \arrow[r, "\Omega\nabla"] \arrow[d] & \Omega(\Sigma A \vee B) \arrow[r, "R"] \arrow[d, "j"] & \Omega(\Sigma A \flat B) \\
\Omega(B, X) \arrow[r, "\exists\nabla'"] \arrow[d, "\partial"'] & \Omega(\Sigma A \vee B, B) \arrow[r, "h"] \arrow[d, "\partial"] & \Omega(\Sigma A \vee B) \arrow[u, "R"'] \\
X \arrow[r] \arrow[d] & B \arrow[d, "i_2"] & \\
B \arrow[r, "\nabla"] & \Sigma A \vee B, &
\end{tikzcd}$$

where the vertical lines are fibrations and ∇' is induced by the commutativity of the bottom square. Then Q' is defined as the composite $R \circ h \circ \nabla'$

Thus the delicate (or crude) Hopf invariants factors through Toda's construction Q' (or Toda's relative Hopf invariants h_T)[13][4]:

$$\begin{tikzcd}
{} \arrow[r, "\Omega i"] & \Omega B \arrow[r, "j"] \arrow[d, "\mathbb{H}"'] & \Omega(B, X) \arrow[r, "\partial"] \arrow[dl, "Q'"'] \arrow[ddl, "h_T"] & X \arrow[r, "i"] & B \\
& \Omega((\Sigma A)\flat B) \arrow[d, "\Omega p'"'] & & & \\
& \Omega^2(\Sigma A \wedge B) & & &
\end{tikzcd}$$

The following theorem is due to T.Ganea[5], this theorem is called as Ganea's cofiber-fiber-cofiber Theorem in [2].

Theorem 2.2. *Given a map $\varphi : A \to X$, consider the cofiber sequence : $A \xrightarrow{\varphi} X \xrightarrow{i} X \cup_\varphi CA = B$.*

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr, "e"'] & & X \arrow[r, "i"] & B \\
& \Omega(B, X) = F_i \arrow[ur, "\partial"'] \arrow[dr, "p"'] & & \\
& & K = C_e &
\end{tikzcd}$$

Then

$$A * \Omega B \xrightarrow{e*1} F_i * \Omega B \xrightarrow{H(\mu)} \Sigma F_i \xrightarrow{\Sigma p} \Sigma K \tag{2}$$

is a homotopy equivalence, where $H(\mu)$ is the Hopf construction of the action $\mu : F_i \times \Omega B \to F_i$ of the principal fibration $F_i \to X \to B$.

So we have

$$\Sigma(\Omega(B, X)) = \Sigma F_i \simeq \Sigma A \vee (A * \Omega B) \simeq \Sigma A \vee (\Sigma A \wedge \Omega B)$$

The rest of this section are devoted to the explanation about Theorem 2.2.

From the above theorem, we get the following:

1. Apply the above theorem for the cofiber seq. $A \to * \to \Sigma A$, then we get the James decomposition Theorem.

$$\begin{aligned}\Sigma\Omega\Sigma A &= \Sigma A \vee \Sigma(A \wedge \Omega\Sigma A)\\ &= \Sigma A \vee (A \wedge \Sigma\Omega\Sigma A)\\ &= \Sigma A \vee (A \wedge (\Sigma A \vee (A \wedge \Sigma\Omega\Sigma A)))\\ &= \Sigma A \vee \Sigma A \wedge A \vee (A \wedge A \wedge \Sigma\Omega\Sigma A)\\ &= \cdots\\ &= \bigvee_{n=1}^{\infty} \Sigma A^{[n]}\end{aligned}$$

Thus we get James-Hopf invariants;

$$h_n : \Omega\Sigma X \to \Omega\Sigma X^{[n]}$$

2. Consider the cofiber sequence $A \xrightarrow{i} X \to X \cup CA \xrightarrow{p} \Sigma A$. The we get

$$\begin{aligned}\Sigma F_p = \Sigma X \vee (X * \Omega\Sigma A) &= \Sigma X \vee (X \wedge \Sigma\Omega\Sigma A)\\ &= \Sigma X \vee (\Sigma X \wedge A) \vee (\Sigma X \wedge A \wedge A) \vee \cdots \vee (\Sigma X \wedge A^{[n]}) \vee \cdots\end{aligned}$$

Thus we have the (Gray) Hopf invariant

$$G_{n+1} : F_p \to \Omega\Sigma(X \wedge A^{[n]}),$$

which was obtained by B.Gray(1972). Gray constructed the relative James model and showed that

$$\begin{tikzcd} \Omega\Sigma A \arrow[r, "h_{n+1}"] \arrow[d, "j"'] & \Omega\Sigma A^{[n+1]} \arrow[d, "\Omega\Sigma(i\wedge 1)"] \\ F_p \arrow[r, "G_{n+1}"'] & \Omega\Sigma(X \wedge A^{[n]}) \end{tikzcd}$$

3. Using the notations in Theorem 2.2, let $J : K \to \Omega(A * \Omega B)$ be the adjoint of the inverse of the homotopy equivalence (2) in Theorem 2.2. Let $\psi : A * \Omega B \to \Omega(\Sigma A \wedge B)$ be the map defined by

$$\psi((1-s)a \oplus s\omega)(t) = [a, t] \wedge \omega(s), \qquad \text{for } a \in A,\ \omega \in \Omega B.$$

Let $Q' : \Omega(B,X) \to \Omega(\Sigma A \flat B)$ be the map which was used for the definition of Toda's relative Hopf invariant. Then there exists a map $T : K \to \Omega(\Sigma A \flat B)$ such that the following diagram commutes (up to homotopy). This result is due to Ganea[5].

$$\begin{array}{ccccc} \Omega(B,X) & \xrightarrow{Q'} & \Omega(\Sigma A \flat B) & \xrightarrow{\Omega p'} & \Omega^2(\Sigma A \wedge B) \\ \| & & \exists T \uparrow & & \uparrow \Omega\psi \\ \Omega(B,X) & \xrightarrow{p} & K & \xrightarrow{J} & \Omega(A * \Omega B) \end{array} \qquad (3)$$

3 Results

Theorem 3.1 (Relative EHP-sequence)**.** *Suppose a map $\varphi : A \to X$ is given. Assume that X is $(m-1)$-connected and A is $(n-2)$-connected. (This implies $\pi_i(X \cup_\varphi CA, X) = 0$ for $i \le n-1$). For simplicity we assume $2 \le m < n$. Let $N = m+n-3+\min\{m, n-1\}$. Then*

1. *The following sequence is a (homotopy) fiber sequence up to dim. N.*

$$\Omega(X \cup_\varphi CA, X) \xrightarrow{p'} \Omega\Sigma A \xrightarrow{H_\varphi} \Omega\Sigma(X \wedge A),$$

Here H_φ is the following composite:

$$\Omega\Sigma A \xrightarrow{h_2} \Omega\Sigma(A \wedge A) = \Omega(A \wedge \Sigma A) \xrightarrow{\Omega(\varphi\wedge 1)} \Omega(X \wedge \Sigma A) = \Omega\Sigma(X \wedge A)$$

(We abbreviate as $B = X \cup_\varphi CA$.)

More precisely, for $i \le N+1$, there exists the following exact sequence of homotopy groups.

$$\pi_i(B,X) \xrightarrow{p'_*} \pi_i(\Sigma A) \xrightarrow{H_\varphi} \pi_i(\Sigma(X \wedge A)) \xrightarrow{\Delta} \pi_{i-1}(B,X) \to \cdots,$$

2. *Boundary homomorphism $\Delta : \pi_i(\Sigma(X \wedge A)) \to \pi_{i-1}(B,X)$ satisfies the following commutative diagram:*

$$\begin{array}{ccccc} \pi_i(\Sigma(X \wedge A)) & \xrightarrow{\Delta} & \pi_{i-1}(B,X) & = & \pi_{i-2}(\Omega(B,X)) \\ E \uparrow \cong & & & & \uparrow w \\ \pi_{i-1}(X \wedge A) & & & & \pi_{i-2}(\Omega X * \Omega^2(B,X)) \\ \| & & & & \uparrow 1*(\Omega e) \\ \pi_{i-2}(\Omega(X \wedge A)) & \underset{\cong}{\xleftarrow{p'}} & \pi_{i-2}(\Omega(X \times A, X \vee A)) & = & \pi_{i-2}(\Omega X * \Omega A), \end{array}$$

Here the map w presents the universal relative Whitehead product, i.,e., $w = [\varepsilon_X, \varepsilon_{\Omega(B,X)}]$, where $\varepsilon_Y : \Sigma\Omega Y \to Y$ is the evaluation map. Recall that $e : A \to \Omega(B,X)$ which carries $a \in A$ to the path $[a,t]$ in the cofiber $B = X \cup_\varphi CA$.

3. *The above exact sequence splits for $i \le 2n-3$, i.e., If $i \le 2n-3$ and $i \le N$, then*

$$\pi_i(B,X) \cong \pi_{i+1}(\Sigma X \wedge A) \oplus \pi_i(\Sigma A),$$

The projection to the first summand is given by Toda's relative Hopf invariant h_T. That is , in the exact sequence, the composite $h_T \circ \Delta$ is an isomorphism:

$$\begin{tikzcd} \pi_{i+1}(\Sigma X \wedge A) \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & \pi_i(B, X) \arrow[r, "p'"] \arrow[d, "h_T"] & \pi_i(\Sigma A) \\ & \pi_{i+1}(\Sigma A \wedge B) & \end{tikzcd}$$

Remark 3.2. The above sequence (1) should be called as "relative EHP sequence". The following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "E"] \arrow[d, "e"'] & \Omega\Sigma A \arrow[r, "h_2"] \arrow[d, equal] & \Omega\Sigma A \wedge A \arrow[d, "\Omega\Sigma(\varphi\wedge 1)"] \\ \Omega(X \cup_\varphi CA, X) \arrow[r, "p'"] \arrow[d, "\partial"'] & \Omega\Sigma A \arrow[r, "H_\varphi"] \arrow[d, "\Omega\Sigma\varphi"'] & \Omega\Sigma(X \wedge A) \arrow[d, "\Omega\Sigma(1\wedge\varphi)"] \\ X \arrow[r, "E"] & \Omega\Sigma X \arrow[r, "h_2"] & \Omega\Sigma X \wedge X, \end{tikzcd}$$

Here the top and bottom lines are so called EHP-sequences.

Note that $\varphi = \partial \circ e$.

Let

$$\lambda_2 : [\Sigma Y, X \cup_\varphi CA] \to [\Sigma^2 Y, (\Sigma A) \wedge (X \cup_\varphi CA)]$$

be the invariant defined by Boardman-Steer(Definition 5.1 in [1]), using the coaction map $\nabla : X \cup CA \to \Sigma A \vee (X \cup CA)$. Then we have

Theorem 3.3. *For a map $f \in [\Sigma Y, X \cup_\varphi CA]$,*

$$\lambda_2(f) = \mathbb{H}'(f)$$

and the following diagram commutes:

$$\begin{tikzcd} C_f \arrow[r, "\bar{\Delta}"] \arrow[d, "p"'] & C_f \wedge C_f \arrow[r] & (C_f/X) \wedge C_f \\ \Sigma^2 Y \arrow[rr, "\lambda_2(f)"] & & (\Sigma A) \wedge (X \cup_\varphi CA). \arrow[u, "i'\wedge i"'] \end{tikzcd} ,$$

where $C_f/X = \Sigma A \cup_{p\circ f} C(\Sigma Y)$ and i' is the natural inclusion map.

The above theorem is a generalization of Theorem 5.14 in Boardman-Steer[1] and clarifies the question in §1 Motivation. Iwase[3] also gives some generalization of Theorem 5.14 in Boardman-Steer[1].

At Halifax conference, 2008, I met Professor H. Marcum and was informed that the commutativity of the above diagram was also obtained by him in 2003 [11].

4 Proofs of theorems

<u>The proof of Theorem 3.1</u> :

We show how to construct the relative EHP sequence. The proof is essentially due to Toda[13], Nomura[12] or Ganea[4, 5].

Consider the following commutative diagram:

$$
\begin{tikzcd}
\Omega^2\Sigma A \arrow[r] \arrow[d, equal] & \Omega F \arrow[r] \arrow[d, "j"] & \Omega(X \cup_\varphi CA) \arrow[r, "p"] \arrow[d, "j"] & \Omega\Sigma A \arrow[r, "\Delta"] \arrow[d, equal] & F \\
\Omega^2\Sigma A \arrow[r] \arrow[d] & G \arrow[r, "k"] \arrow[d] & \Omega(X \cup_\varphi CA, X) \arrow[r, "p'"] \arrow[d, "\partial"] & \Omega\Sigma A \arrow[d] & \\
* \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d, "i'"] & X \arrow[r] \arrow[d, "i"] & * \arrow[d] & \\
\Omega\Sigma A \arrow[r] & F \arrow[r] & X \cup_\varphi CA \arrow[r, "p"] & \Sigma A, &
\end{tikzcd}
$$

where G is the homotopy fiber of i', that is $G = \Omega(F, X)$.

James exact sequence can be obtained by the fiber sequence of the second line from the top in the above diagram. In fact, there exists a map $h : (F, X) \to (\Omega\Sigma(X \wedge A), *)$, which makes the following diagram commute up to homotopy:

$$
\begin{tikzcd}
 & & \Omega(F, X) = G \arrow[dl, "h"] \\
\Omega^2\Sigma A \arrow[r] \arrow[urr] \arrow[d, "h_2"] & \Omega F \arrow[ur, "j"] \arrow[d, "G_2"] & \\
\Omega^2\Sigma A \wedge A \arrow[r, "\Omega^2\Sigma(\varphi\wedge 1)"] & \Omega^2\Sigma X \wedge A, &
\end{tikzcd}
$$

where G_2 is the map in 2 in §2. By using Blakers-Massey Theorem and approximating F by the second stage F_2 of the Gray model, we see that the above h induces an isomorphism between the homotopy groups $\pi_i(F, X) \to \pi_i(\Omega\Sigma(X \wedge A))$ for $i \leq N = m + n - 3 + \min\{m, n - 1\}$.

Therefore, delooping once, we obtain the desired sequence. This proves 1 in Theorem 3.1.

The boundary homomorphism $\Delta : \pi_{i+1}(\Sigma X \wedge A) \to \pi_i(X \cup_\varphi CA, X)$ of the relative EHP is related to the relative Whitehead products as in the following manner.

Consider the following diagram:

$$\begin{tikzcd}
\Omega X * \Omega A \arrow[r, "1*\Omega e"] \arrow[d] & \Omega X * \Omega(\Omega(X \cup_\varphi CA, X)) \arrow[rr, "\exists W"] \arrow[d] & & G \arrow[d] \\
X \vee A \arrow[r, "1\vee e"] \arrow[d] & X \vee \Omega(X \cup_\varphi CA, X) \arrow[r, "1_X \vee \partial"] \arrow[d] & X \vee X \arrow[r, "fold"] & X \arrow[d] \\
X \times A \arrow[r, "1\times e"] & X \times \Omega(X \cup_\varphi CA, X) \arrow[r, "i' \times \Omega p'"] & F \times \Omega\Sigma A \arrow[r, "\mu"] & F,
\end{tikzcd}$$

where the 3 vertical lines are fiber sequences.

It is known that W is N-connected and that the following composite w gives the universal relative Whitehead product:

$$w : \Omega X * \Omega(\Omega(X \cup_\varphi CA, X)) \xrightarrow{W} G \xrightarrow{k} \Omega(X \cup_\varphi CA, X)$$

The bottom square in the above diagram induces the bottom inclusion map $i : X \wedge A \to F/X$ which follows Gray's relative Hopf construction, and using the top diagram this induces the following commutative diagram :

$$\begin{tikzcd}
 & \Omega^2\Sigma X \wedge A & \\
\Omega(X \wedge A) \arrow[r, "\Omega i"] \arrow[ur, "\Omega E"] & \Omega(F/X) \arrow[u, "h"'] & \\
\Omega X * \Omega A \arrow[u, "p'"'] \arrow[r, "W\circ 1*(\Omega e)"] \arrow[dr, "1*\Omega e"'] & G = \Omega(F, X) \arrow[u, "p'"'] \arrow[r, "k"] & \Omega(X \cup CA, X) \\
 & \Omega X * \Omega^2(X \cup CA, X) \arrow[u, "W"'] \arrow[ur, "w"'] &
\end{tikzcd}$$

Since in our range of dimension, all the maps except k and w, induces the isomorphisms of homotopy groups. These gives the proof of 2 in Theorem 3.1.

Remark that Nomura[12] showed the following fact:

If $i = m+n-2$ then $\pi_i(\Omega X * \Omega(\Omega(X \cup_\varphi CA, X)))$ is generated by $\pi_{m-1}(\Omega X)\hat{*}((\Omega e)_*(\pi_{n-2}(\Omega A))$, where $e : A \to \Omega(X \cup_\varphi CA, X)$ is a natural map described previously and $\hat{*} : \pi_p(U) \times \pi_q(V) \to \pi_{p+q+1}(U * V)$ is the join operation. In the case that $A = S^{n-1}$, e represents the characteristic map of this cell.

Now we give the proof of 3 of Theorem 3.1.

Let

$$\Omega^2(B, X) \xrightarrow{j} \Omega(\Omega(B, X), A) \xrightarrow{\partial} A \xrightarrow{e} \Omega(B, X)$$

be the fiber sequence. Then by diagram chasing, we see that there exists the following fiber sequence:

$$\Omega G = \Omega^2(F, X) \xrightarrow{j\circ\Omega k} \Omega(\Omega(B, X), A) \xrightarrow{p'} \Omega(\Omega\Sigma A, A).$$

Since A is $(n-2)$ connected, by suspension theorem, the first map in the above sequence induces the isomorphism of the homotopy groups for our range. On the other hand, Toda's relative Hopf invariant factors through as the diagram (3), we obtain $\Omega h_T \circ \Omega k = \Omega^2\psi \circ \Omega J \circ p' \circ (j \circ \Omega k)$

$$\begin{tikzcd}
 & \Omega(\Omega(B,X),A) \arrow[dr, "p'"] & \\
\Omega G \arrow[ur, "j\circ\Omega k"] \arrow[r, "\Omega k"] & \Omega^2(B,X) \arrow[u, "j"] \arrow[r, "\Omega p"] \arrow[d, "\Omega h_T"] & \Omega K \arrow[d, "\Omega J"] \\
 & \Omega^3(\Sigma A \wedge B) & \Omega^2\Sigma A \wedge B \arrow[l, "\Omega^2\psi"']
\end{tikzcd}$$

From the assumption about the connectivity, we see that all the maps $\Omega^2\psi$, ΩJ, p' and $j \circ \Omega k$ induces the isomorphisms of homotopy groups. This proves 3 of Theorem 3.1.

<u>The proof of Theorem 3.3</u> :

We follow after the proof of Boardman-Steer. First we see that the Boardman-Steer construction $\mu : [\Sigma Y, B_1 \vee B_2] \to [\Sigma^2 Y, B_1 \wedge B_2]$ factors as follows:

$$\begin{array}{ccc}
[\Sigma Y, B_1 \vee B_2] & \xrightarrow{\mu} & [\Sigma^2 Y, B_1 \wedge B_2] \\
\mu' \big\downarrow & & \cong \big\downarrow adj \\
[\Sigma Y, \Omega(B_1 \times B_2, B_1 \vee B_2)] & \xrightarrow{p'_*} & [\Sigma Y, \Omega(B_1 \wedge B_2)]
\end{array}$$

We will explain the above diagram:

Let T is the triangle in I^2 of points (s,t) with $s \le t$. Given a map $g : \Sigma Y \to B_1 \vee B_2$, Boardman-Steer constructs a map $q(g) : (Y \times T, Y \times \partial T, Y \times (0,1) \cup y_0 \times T) \to (B_1 \times B_2, B_1 \vee B_2, *)$ by the formula $q(g)(y,s,t) = ((p_1 \circ g)(y,s), (p_2 \circ g)(y,t)) \in B_1 \times B_2$. Fixing a base point preserving homeomorphism $(T, \partial T, (0,1)) \to (CS^1, S^1, *)$, we see that this q induces canonically a map $\mu'(g) : (Y \wedge CS^1, Y \wedge S^1) \to (B_1 \times B_2, B_1 \vee B_2)$. Note that $\mu'(g)$ can be seen as a map from ΣY to $\Omega(B_1 \times B_2, B_1 \vee B_2)$ canonically. By the construction of μ' (see Boardman Steer p201), we see that

$$(\partial \circ \mu')(g) = -i_2 \circ p_2 \circ g + g - i_1 \circ p_1 \circ g, \tag{4}$$

where $\partial : \Omega(B_1 \times B_2, B_1 \vee B_2) \to B_1 \vee B_2$ is the fiber inclusion and $i_j : B_j \to B_1 \vee B_2$ is the inclusion to the j-th factor. Therefore μ' is equal to the adjoint of $R : \Omega(B_1 \vee B_2) \to \Omega(\Omega(B_1 \times B_2, B_1 \vee B_2))$. Now, given a map $f : \Sigma Y \to X \cup CA$, let $g = \nabla f : \Sigma Y \to \Sigma A \vee X \cup CA$. By definition $\lambda_2(f)$ is equal to the adjoint of $p'_*(\mu'(g))$ and by definition $\mathbb{H}'(f) = \Omega p' \circ R \circ \Omega\nabla \circ \mathrm{adj}.f$. This proves the first assertion $\lambda_2(f) = \mathbb{H}'(f)$.

Remark that in the equation (4) the order of the three elements g, $-i_1 \circ p_1 \circ g$ and $-i_2 \circ p_2 \circ g$ and the choice of sign depends on the choice of the base point of T and the choice of its orientation.

Next observe that various homotopy in Boardman-Steer p202-p203 can be modified to fit in our case. First we need the homotopies $g_u : \Sigma C \to \Sigma C$ and $k_u : \Sigma C \to \Sigma C$ for any suspension space

ΣC, defined by the formula

$$g_u(c,t) = \begin{cases} (c, t(1+u)) & \text{for } 0 \le t \le \dfrac{1}{1+u} \\ * & \text{for } \dfrac{1}{1+u} \le t \le 1 \end{cases}$$

$$k_u(c,t) = \begin{cases} * & \text{for } 0 \le t \le \dfrac{u}{1+u} \\ (c, t(1+u)-u) & \text{for } \dfrac{u}{1+u} \le t \le 1. \end{cases}$$

Similarly, for the mapping cone $B = X \cup CA$, we need the homotopy $G_u : B \to B$ defined by

$$G_u(x) = x \quad \text{for} \qquad \forall x \in X$$

$$G_u(a,t) = \begin{cases} (a, t(1+u)) & \text{for } 0 \le t \le \dfrac{1}{1+u} \\ * & \text{for } \dfrac{1}{1+u} \le t \le 1, \end{cases} \qquad \text{for } (a,t) \in CA$$

As is well-known, the modified diagonal map $\bar{\Delta}_B : B \to \Sigma A \wedge B$ defined by $\bar{\Delta}(y) = p(y) \wedge y$, is null-homotopic. Then $(k_u \wedge G_u) \circ \bar{\Delta}_B : B \to \Sigma A \wedge B$ is the null-homotopy of it.

Given a map $f : \Sigma Y \to X \cup CA = B$, let $C_f = B \cup C\Sigma Y$ be the mapping cone. We will construct a homotopy $F_u : C_f \to C_{p\circ f} \wedge C_f$ in the three stages from $F_0 : C_f \xrightarrow{\bar{\Delta}} C_f \wedge C_f \to C_{p\circ f} \wedge C_f$.

First stage: $0 \le u \le 1$.

On B we take the constant homotopy,

$$F_u = \bar{\Delta}_B : B \to \Sigma A \wedge B \subseteq C_{p\circ f} \wedge C_f.$$

On $C\Sigma Y$, for $z \in \Sigma Y$, we take

$$F_u(z,t) = \begin{cases} (k_{2tu} z, t) \wedge (g_{2tu} z, t) & \text{for } 0 \le t \le 1/2 \\ (k_u z, t) \wedge (g_u z, t) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Note that F_1=zero on the points (z,t) in $C\Sigma Y$ for $1/2 \le t \le 1$ and $z \in \Sigma Y$.

Second stage: $1 \le u \le 2$.

Since the image of F_1 lies in $M \wedge M$, where M is the subset of $B \cup C\Sigma Y$ corresponding to the set $B \cup (\Sigma Y \times [0, 1/2])$. Since M contains B as a canonical deformation retract, we find $F_2 : C_f \to C_{p\circ f} \wedge C_f$:

On B, $F_2 = \bar{\Delta}_B$, still

On $C\Sigma Y$

$$F_2(z,t) = \begin{cases} (p(f(k_{2t} z)) \wedge f(g_{2t} z) \in \Sigma A \wedge B \subseteq C_{p\circ f} \wedge C_f & \text{for } 0 \le t \le 1/2 \\ * & \text{for } 1/2 \le t \le 1. \end{cases}$$

Third stage: $2 \le u \le 3$.

Define the homotopy for $2 \le u \le 3$ by:

$$C_f \xrightarrow{F_2} \Sigma A \wedge B \xrightarrow{k_{u-2} \wedge G_{u-2}} \Sigma A \wedge B.$$

Then F_3 is zero except on the set of the points (z,t) in $C\Sigma Y$ for $0 \le t \le 1/2$ and $z \in \Sigma Y$, on which we have

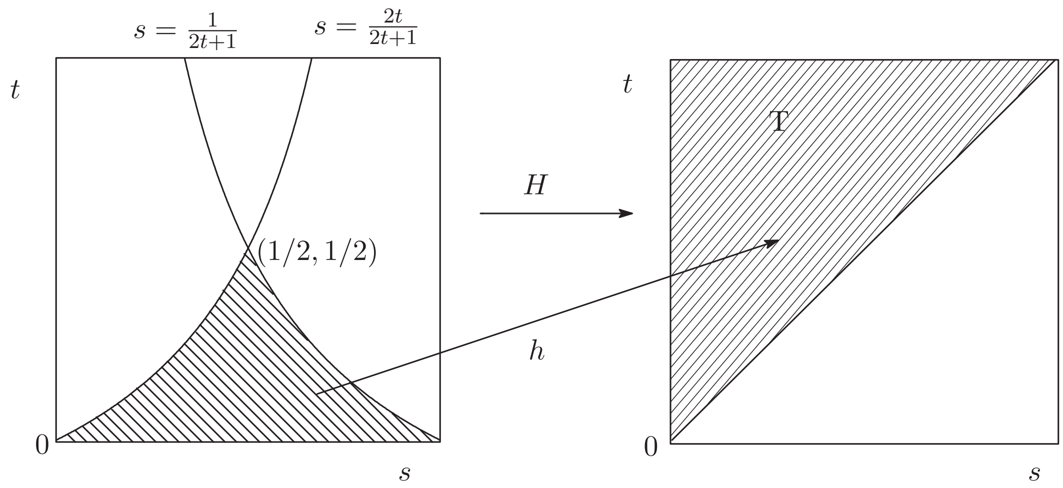
$$F_3(z,t) = k_1(p(f(k_{2t} z)) \wedge G_1(f(g_{2t} z)), \quad \text{for } (z \in \Sigma Y, t \in [0, 1/2]).$$

Now let $\nabla : B \to \Sigma A \vee B$ be the coaction map. Note that the composite $B \xrightarrow{\nabla} \Sigma A \vee B \subseteq \Sigma A \times B$ is just equal to the map $(k_1 \circ p) \times G_1$. Thus $p_1 \circ (\nabla \circ f) = k_1(p(f)) \circ f$ and $p_2 \circ (\nabla \circ f) = G_1 \circ f$. If we compare F_3 with μ-construction $\mu(\nabla \circ f)$, we see that F_3 factors through $\Sigma^2 Y$ by the map $F : \Sigma^2 Y \to \Sigma A \wedge B$ defined by $F(y, s, t) = k_1(p(f(g_{2t}(y, s)) \wedge G_1(f(k_{2t}(y, s)))$ for $(y, s, t) \in Y \times I \times I$. And by definition of $\mu(\nabla f)$

$$F(y,s,t) = \begin{cases} (p_1 \circ \nabla f)(y, s(1+2t) - 2t) \wedge (p_2 \circ \nabla f)(y, s(2t+1)) \\ \qquad\qquad\qquad\qquad \text{for } \dfrac{2t}{2t+1} \leq s \leq \dfrac{1}{2t+1} \\ * \qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

Thus we see that F factors as $F = \mu(\nabla f) \circ H$, where $H : \Sigma^2 Y \to \Sigma^2 Y$ is induced by a map $h(y, s, t) = (y, s(1+2t) - 2t, s(2t+1))$ on the region of $\dfrac{2t}{2t+1} \leq s \leq \dfrac{1}{2t+1}$ as in the figure below. (Clearly there exists an extension of h, which induces the map $I^2/\partial I^2 \to I^2/\partial I^2$ homotopic to identity. This extension gives a map H). Thus it follows that F_3 and $\mu(\nabla f)$ is homotopic.

This completes the proof.



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