Definable slices

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Abstract

Let G be a definably compact definable group and X a definable G set. We prove that there exists a definable slice at every point of X and X is covered by finitely many definable G tubes.

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1 . Introduction.

Let G be a topological group, X a G space and $x \in X$. A *slice* at x is a subset S of X containing x such that $G_x S = S$ and the map $\phi : G \times_{G_x} S \to X$ defined by $\phi([g, s]) = gs$ is a G imbedding onto a G invariant open neighborhood GS of $G(x)$ in X, and GS is called a *G tube*. The existence of a slice when G is a compact Lie group and X is a completely regular G space is studied ([4], [10], [11]).

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Everything is considered in $\mathcal{N}$ and each definable map is assumed to be continuous unless otherwise stated.

General references on o-minimal structures are [2], [3], also see [14].

Let G be a definable group. A pair (X, ϕ) consisting a definable set X and a G action $\phi : G \times X \to X$ is a *definable G set* if ϕ is definable. We simply write X instead of (X, ϕ).

In this paper we prove the existence of a slice in the definable category.

Theorem 1.1. *Let G be a definably compact definable group and X a definable G set.*

(1) For every point $x \in X$, there exists a definable slice S at x.

(2) X is covered by finitely many definable G tubes.

Theorem 1.1 is a generalization of [6].

Let $GL(n, R)$ be the set of invertible $n \times n$ matrices over R. Then $GL(n, R)$ is a definable group, and we call it the nth *general linear group*. A definable subgroup of some $GL(n, R)$ is a *definable linear group*.

If $\mathcal{N}$ is an o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots,)$ of the field $\mathbb{R}$ of real numbers, then we have the following result.

Theorem 1.2. *If $\mathcal{N} = \mathcal{M}$ and G is a compact definable linear group, then every definable G set is definably G imbeddable into* *some definable orthogonal G representation space.*

2 . Preliminaries.

For every $a, b \in R \cup \{\infty\} \cup \{-\infty\}$ with $a < b$, let $(a, b)_R$ denote $\{x \in R | a < x < b\}$. For any $a, b \in R$ with $a < b$, let $[a, b]_R$ denote $\{x \in R | a \leq x \leq b\}$.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset R^n \times R^m$) is a definable set. A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.

A definable map between definable G sets is a *definable G map* if it is a G map. A definable G map is a *definable G homeomorphism* if it is a homeomorphism.

A definable set X is *definably compact* if for every $a, b \in R \cup \{\infty\} \cup \{-\infty\}$ with $a < b$ and for every definable map $f : (a, b)_R \to X$, $\lim_{x \to a+0} f(x)$ and $\lim_{x \to b-0} f(x)$ exist in X.

If $R = \mathbb{R}$, then for any definable subset X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general a definably compact definable set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}}$ is definably compact but not compact.

Theorem 2.1. *(1) (Monotonicity (e.g. 3.1.2, 3.1.6 [2])). Let $f : (a, b)_R \to R$ be a function with the definable graph. Then there exist points $a = a_0 < a_1 < \cdots < a_k = b$ in $(a, b)_R$ such that for each j with $0 \leq j \leq k - 1$, $f | (a_j, a_{j+1})_R$ is constant, or strictly monotone and continuous. Moreover for each $c \in (a, b)_R$, $\lim_{x \to c+0} f(x)$ and $\lim_{x \to c-0} f(x)$ exist in $R \cup \{\infty\} \cup \{-\infty\}$.*
(2) (Definable triangulation (e.g. (8.2.9 [2])). Let $S \subset R^n$ be a definable set and $S_1, \dots, S_k$ definable subsets of S. Then there exist a finite simplicial complex K in R^n and a definable map $\phi : S \to R^n$ such that ϕ maps S and each S_i definably homeomorphically onto a union of open simplexes of K. If S is definably compact, then we can take $K = \phi(S)$.
(3) (Piecewise definable trivialization (e.g. 9.1.2 [2])). Let X and Y be definable sets and $f : X \to Y$ a definable map. Then there exist a finite partition $\{T_i\}_{i=1}^k$ of Y into definable sets and definable homeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(y_i)$ such that $f | f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \leq i \leq k)$, where $y_i \in T_i$ and $p_i : T_i \times f^{-1}(y_i) \to T_i$ denotes the projection.
(4) (Existence of definable quotient (e.g. 10. 2.18 [2])). Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.

Recall the definition of orbit types ([6], [5], [8]). Let G be a definably compact definable group. We say that two homogeneous definable G sets are *equivalent* if they are definably G homeomorphic. Let (G/H) be the equivalence class of G/H. The set of equivalence classes of homogeneous definable G sets has an order defined as $(X) \geqq (Y)$ if there exists a definable G map $X \to Y$. Then the reflexivity and the transitivity hold and the anti-symmetry is true ([6], [5], [8]).

By a way similar to the proof of 1.3 [6], we have the following theorem.

Theorem 2.2. *Let G be a definably compact definable group. Then every definable G set has only finitely many orbit types.*

Theorem 2.3. *Let G be a definably compact definable group, X a definable G set with transitive action and $x \in X$. Then the map $f : G/G_x \to X$ defined by $f(gG_x) = gx$ is a definable G homeomorphism.*

Proof. Since the isotropy subgroup G_x of x is a definable (closed) subgroup of G, G_x is definably compact. By Theorem 2.1, G/G_x exists as a definable set. By the proof of 1.5 [9], f is a bijective G map, and f is definable because f is induced by a definable map $G \to X, g \mapsto gx$. Since G is definably compact, f is a definable G homeomorphism. □

3 . Definable slices.

Let G be a definably compact definable group, X a definable G set and $x \in X$. A *definable slice* at x is a definable subset S of X containing x such that $G_x S = S$ and the map $\phi : G \times_{G_x} S \to X$ defined by $\phi([g, s]) = gs$ is a definable G imbedding onto a G invariant definable open neighborhood GS of $G(x)$ in X, and GS is called a *definable G tube*. Remark that $G \times_{G_x} S$ exists a definable set because G_x is definably compact and Theorem 2.1, and the natural G action $G \times G \times_{G_x} S \to G \times_{G_x} S, (g, [g', x]) \mapsto [gg', x]$ induced by $G \times G \times S \to G \times S, (g, (g', x)) \mapsto (gg', x)$ is definable.

Proposition 3.1 (e.g. II. 4.2 [1]). *Let G be a compact Lie group, X a G set, S a subset of X and $x \in S$. Then the following three conditions are equivalent.*

(1) There exists a G imbedding $\phi : G \times_{G_x} A \to X$ onto a G invariant open neighborhood of $G(x)$ with $\phi([e, A]) = S$, where A is a G_x space.

(2) S is a slice at x.

(3) GS is a G invariant open neighborhood of $G(x)$ and there exists a G retraction $f : GS \to G(x)$ such that $f^{-1}(x) = S$.

By a way similar to the proof of Proposition 3.1, we have the following proposition.

Proposition 3.2. *Let G be a definably compact definable group, X a definable G set, S a definable subset of X and $x \in S$. Then the following three conditions are equivalent.*

(1) There exists a definable G imbedding $\phi : G \times_{G_x} A \to X$ onto a G invariant definable open neighborhood of $G(x)$ with $\phi([e, A]) = S$, where A is a definable G_x set.

(2) S is a definable slice at x.

(3) GS is a G invariant definable open neighborhood of $G(x)$ and there exists a definable G retraction $f : GS \to G(x)$ such that $f^{-1}(x) = S$.

Proposition 3.3. *Let G be a definably compact definable group and S a definable slice at x in a definable G set X. Then the map $f : S/G_x \to X/G$ defined by $[s] \mapsto [s]$ is a definable homeomorphism onto the G invariant definable open subset GS/G.*

Proof. By a fact in topological group theory (see II.4.7 [1]), f is a homeomorphism. Since f is induced by $S \to GS, s \mapsto s$, f is definable. □

Let G be a definable group. Let f be a G invariant surjective definable map from a definable G set X to a definable set Y. We say that f is *definably G trivial* if there exists a definable G homeomprphism $k : X \to Y \times f^{-1}(a)$ with $f = p \circ k$, where $a \in X$ and p denotes the projection $Y \times f^{-1}(a) \to Y$.

By a way similar to the proof of 2.5 [7], we have the following theorem.

Theorem 3.4. *Let G be a definably compact definable group, X a definable G set, Y a definable set and $f : X \to Y$ a G invariant surjective definable map. Then there exists a finite partition $\{C_i\}_i$ of Y into definable sets such that each $f|f^{-1}(C_i) : f^{-1}(C_i) \to C_i$ is definably G trivial.*

A way similar to the proof of 4.3 [6], we have the following lemma.

Lemma 3.5. *Let X be a definable set and $f : X \to R$ (resp. $g : X \to R$) a lower (resp. upper) semi-continuous function such that they have definable graphs and $g(x) \le f(x)$ for all $x \in X$. Then there exists a definable function $h : X \to R$ such that $g(x) \le h(x) \le f(x)$ for all $x \in X$ and $g(x) < h(x) < f(x)$ whenever $g(x) < f(x)$.*

Proposition 3.6. *Let X be a definable set and A a definable closed subset of X. Suppose that A is a definable strong deformation retract of X. Then for any definable open neighborhood U of A in X, there exist a definable closed neighborhood N of A in U and a definable map $\rho : X \to U$ such that $\rho|N = id$ and $\rho(X - N) \subset U - N$.*

Proof. Let $F : X \times [0, 1]_R \to X$ be a definable strong deformation retraction from X to A. Let $g : X \to [0, 1]_R$ be the function defined by $g(x) = \inf\{r \in [0, 1]_R | F(x, t) \in U$ for all $t \in (r, 1]_R\}$. Then g has the definable graph. We now prove that g is upper semi-continuous. We need to show that for every $a \in R$, $\{x \in X | g(x) < a\}$ is open. For x_0 with $g(x_0) < a$, take b such that $g(x_0) < b < a$. By the definition of g, $F(x_0, t) \in U$ for all $t \in [b, 1]_R$. We define a function $\phi : [b, 1]_R \to R, \phi(t) = \min\{\sup\{t' > 0 | F(N(x_0; t'), t) \subset U\}, 1\}$, where $N(x_0; t')$ denotes the definable open t' neighborhood of x_0 in X. Then ϕ is a positive function with the definable graph. By Theorem 2.1, there exist points $b = b_0 < b_1 < \cdots < b_k = 1$ in $[b, 1]_R$ such that for each j with $0 \le j \le k-1$, $\phi|(b_j, b_{j+1})_R$ is constant, or strictly monotone and continuous. Moreover $\lim_{x \to b_j+0} \phi(x)$ and $\lim_{x \to b_j-0} \phi(x)$ exist in R. By construction of ϕ, $\lim_{x \to b_j+0} \phi(x)$, $\lim_{x \to b_j-0} \phi(x)$ are positive. Thus modifying ϕ, if necessary, we may assume that for each j with $0 \le j \le k-1$, $\phi|[b_j, b_{j+1}]_R$ is a positive definable function. Since $[b_j, b_{j+1}]_R$ is definably compact, $\phi|[b_j, b_{j+1}]_R$ has the minimum $\epsilon_j > 0$. Let $\epsilon = \min\{\min_j \epsilon_j, \min_j \phi(b_j)\} > 0$ and $V = N(x_0; \epsilon)$. Then $F(V \times [b, 1]_R) \subset U$. Since $g(y) \le b < a$, $g^{-1}(\{y < a\})$ is open. Hence g is upper semi-continuous.

Since $F(A \times [0, 1]_R) = A \subset U$ and by the above argument, for any $a_0 \in A$, there exists an $\epsilon_{a_0} > 0$ such that $F(N(a_0; \epsilon_{a_0}) \times [0, 1]_R) \subset U$. Replacing ϵ_{a_0} by $\frac{\epsilon_{a_0}}{2}$, we may assume that $F(\overline{N(a_0; \epsilon_{a_0})} \times [0, 1]_R) \subset U$, where $\overline{N(a_0; \epsilon_{a_0})}$ denotes the closure of $N(a_0; \epsilon_{a_0})$ in X. We define a function $\epsilon : A \to R, \epsilon(a) = \min\{\frac{1}{2} \sup\{\epsilon' > 0 | F(\overline{N(a; \epsilon')} \times [0, 1]_R) \subset U\}, 1\}$. Then ϵ is a positive function with the definable graph.

Let $N = \overline{\cup_{a_0 \in A} N(a_0; \epsilon(a_0))}$. Then N is a definable closed neighborhood N of A such that $F(N \times [0, 1]_R) \subset U$. Let $f : X \to [0, 1]_R$ be the function defined by $f(x) = \inf\{r \in [g(x), 1]_R | F(x, r) \in N\}$. Then f is well defined, it has the definable graph, $g(x) = f(x) = 0$ for all $x \in N$ and $g(x) < f(x)$ for all $x \notin N$.

We now prove that f is lower semi-continuous. Let $x_0 \notin N$ and take a with $g(x_0) < a < f(x_0)$. Choose $b, c \in [0, 1]_R$ such that $g(x_0) < b < a < c < f(x_0)$. Since g is upper semi-continuous, there exists a definable open neighborhood V of x_0 such that $g(x) < b$ whenever $x \in V$. Since N is closed and $[b, c]_R$ is definably compact and by the above argument, there exists a neighborhood V' of x_0 such that $F(V' \times [b, c]_R) \cap N = \emptyset$. This implies that if $x \in V'$ then $f(x) > a$. Hence f is lower semi-continuous on $X - N$. Since $f|N = 0$, f is lower semi-continuous on X.

By Lemma 3.5, there exists definable function h such that $g(x) \le h(x) \le g(x)$ for all $x \in X$ and the inequalities become strict whenever $g(x) \ne f(x)$. Let $\rho(x) = F(x, h(x))$. Then $\rho(x) = F(x, 0) = x$ for all N and if $x \notin N$ then $\rho(x) = F(x, h(x)) \in U - N$ because $g(x) < h(x) < f(x)$. □

Proof of Theorem 1.1. By Theorem 2.2, X has finitely many orbit types. Let $(G/H_1), \dots, (G/H_k)$ be the orbit types. Then for each i, X^{H_i} is a definable $N(H_i)$ set, where $N(H_i)$ denotes the normalizer of H_i in G. Applying Theorem 2.1 to the orbit map $\pi_{H_i} : X^{H_i} \to X^{H_i}/N(H_i)$, there exist a finite partition $\{T_{i_j}\}$ and definable sections $s_{i_j} : T_{i_j} \to X^{H_i}$. Using Theorem 2.1, we take a definable triangulation (K, ϕ) of X/G compatible with $\{T_{i_j}\}$. Replacing K by its subdivision, we may assume that K contains $\pi(x)$ as a 0-simplex,

(A) every $\Delta \in K$ contains a 0-simplex, and

(B) the interior Int Δ of Δ has a definable section $s : \mathrm{Int}\,\Delta \to X$ of $\pi : X \to X/G$ such that $s(\mathrm{Int}\,\Delta)$ has a constant stabilizer.

Let $\{v_0 = \pi(x_0), v_1, \dots, v_l\}$ be the set of vertices of K. By (A), the open star neighborhoods $\{St(v_i)\}_{i=1}^l$ is an open cover of $|K| = X/G$ and $\{\pi^{-1}(St(v_i))\}_{i=1}^l$ is an open cover of X . We claim that for any vertex v, $\pi^{-1}(St(v))$ is a definable G tube of the orbit $\pi^{-1}(v)$. By Proposition 3.2, it is enough to construct a definable G retraction $f : \pi^{-1}(St(v)) \to \pi^{-1}(v)$. By the induction on n, we now construct a definable G retraction $f_n : \pi^{-1}(St(v)^n) \to \pi^{-1}(St(v)^{n-1})$ for each n. Then the composition $f = f_1 \circ \cdots \circ f_n$ is the required G retraction.

Let Δ be an n-simplex of K containing v as a vertex and let $\Delta^n = \Delta \cap St(v)^{(n)}$. Note that Δ is closed in K. Since each n-simplex of $St(v)$ is of the form Δ^n, we restrict our attention to construct a definable G retraction $f_n : V = \pi^{-1}(\Delta^n) \to \pi^{-1}(\partial\Delta^n) = \partial V$, where $\partial\Delta^n = \Delta \cap St(v)^{(n-1)}$.

Since $\Delta^n - \partial\Delta^n = \mathrm{Int}\,\Delta$ and by (B), there exists a definable section $s : \Delta^n - \partial\Delta^n \to X^H \subset X$, where H is some H_i. Let W be the closure of $s(\Delta^n - \partial\Delta^n)$ in V and we simply write $\partial W = W \cap \partial V$. We claim that there exists a definable retraction $\tilde{r} : W \to \partial W$. Let $\tilde{U}$ be a definable regular neighborhood of ∂W in W. Then $U = \pi(\tilde{U})$ is a definable neighborhood of $\partial\Delta^n$. By a suitable definable homeomorphism, the pair $(\Delta^n, \partial\Delta^n)$ is definably homeomorphic to a pair composed of a simplex and one of its faces (Δ, Δ^{n-1}) with a neighborhood which is definably homeomorphic to U. By Proposition 3.6, there exist a definable closed neighborhood $N \subset U$ and a definable retraction $r : \Delta \to N$ such that $r(\Delta - \Delta^{n-1}) \subset N - \Delta^{n-1}$. We define a definable map $r' : W \to \tilde{U}, r'(x) = \begin{cases} s \circ r \circ \pi(x), & x \in W - \partial W \\ x, & x \in \partial W \end{cases}$.
Since the regular neighborhood $\tilde{U}$ has a definable retraction to ∂W, the composition of this map and r' gives a definable retraction $\tilde{r} : W \to \partial W$.

The map $f_n : V = GW \to G(\partial W) = \partial V$ defined by $f_n(gx) = g\tilde{r}(x)$ is the required definable G retraction. □

4 . Definable G imbeddings.

In this section we assume that $\mathcal{N}$ is an o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ of the field $\mathbb{R}$ of real numbers.

Let G, G' be definable groups. A group homomorphism between G and G' is a *definable group homomorphism* if it is definable. A *definable G representation* is a definable group homomorphism $\phi : G \to GL(n, \mathbb{R})$ for some n. A *definable G orthogonal representation* is a definable group homomorphism $\phi : G \to O(n)$ for some n. In this case $\mathbb{R}^n$ with the orthogonal action of G via ϕ is denoted by $\mathbb{R}^n(\phi)$ and called a *definable orthogonal G representation space.*

Lemma 4.1. *Every compact subgroup H of a definable linear group G is a definable subgroup.*

Proof. Since G is a definable linear group, G is a definable subgroup of $GL(n, \mathbb{R})$. Then H is a compact Lie subgroup of $GL(n, \mathbb{R})$ because $GL(n, \mathbb{R})$ is a Lie group. Let $M(n, \mathbb{R})$ be the set of $n \times n$ matrices over $\mathbb{R}$. Then $M(n, \mathbb{R})$ is an H representation space whose H action is defined by the matrix left multiplication. Every H orbit of $M(n, \mathbb{R})$ is the inverse image of one point in the orbit space by the orbit map $\pi : M(n, \mathbb{R}) \to M(n, \mathbb{R})/H$. Since H is a compact Lie group, π is a polynomial map. Thus every H orbit is definable. Since H is an orbit of the identity matrix, H is definable. □

Corollary 4.2. *If $f : G \to G'$ is a topological group homomorphism between definable linear groups G, G' and G is compact, then f is definable.*

Proof. By the assumption, the graph $\Gamma(f)$ of f is a compact subgroup of the definable linear group $G \times G'$. Thus f is definable by Lemma 4.1. □

Proposition 4.3. *Let G be a compact definable linear group and H a definable (closed) subgroup of G. Then there exist a definable faithful representation $\phi : G \to O(n)$ for some n and a point $v \in \mathbb{R}^n(\phi)$ such that $v \neq 0, G_v = H$.*

Proof. Since G is a compact definable linear group, G is a compact subgroup of $GL(n, \mathbb{R})$. Hence G is a compact Lie group. By the theory of compact Lie groups, there exist a faithful representation $\phi : G \to O(n)$ for some n and a point $v \in \mathbb{R}^n(\phi)$ such that $G_v = H$. By Corollary 4.2, ϕ is a definable homomorphism. □

By Corollary 4.2 and facts in topological group theory (see 1.4.2 [12]), we have the following proposition.

Proposition 4.4. *Let G be a compact definable linear group and H a definable* *(closed) subgroup of G. If Ω is a definable orthogonal H representation space, then there exists a definable orthogonal G representation space Ξ such that considering Ξ as an H space by restriction, Ξ has Ω as an H invariant linear subspace.*

Let G be a comapct definable group, X a definable G set and H a definable subgroup of G. Note that H is a closed subgroup ([13]). A definable subset S of X is a *definable H kernel* if there exists a definable G map $f : GS \to G/H$ such that $f^{-1}(eH) = S$. Note that by Theorem 2.3 and Proposition 3.2, every slice at x is a definable G_x kernel.

Proposition 4.5. *Let G be a compact definable linear group and H a definable (closed) subgroup of G. If Ω is a definable orthogonal H representation space, then there exists a definable H imbedding of Ω onto a definable H kernel in some orthogonal definable G representation space Ξ.*

Proof. By Proposition 4.3, there exist a definable orthogonal G representation space Ξ' and a point $u_0 \in \Xi'$ such that $u_0 \neq 0, G_{u_0} = H$. By Proposition 4.4, there exists a definable orthogonal G representation space Ω' including Ω as an H invariant linear space. Let $\Xi = \Xi' \oplus \Omega'$. Then Ξ is a definable orthogonal G representation space and $\phi : \Omega \to \Xi = \Xi' \oplus \Omega', \phi(v) = (u_0, v)$ is a definable H imbedding. Moreover $S = \phi(\Omega)$ is an H invariant definable closed subset of Ξ. If $g \notin H$ and $(u_0, v) \in S$, then $g(u_0, v) \notin S$ because $g \notin H = G_{u_0}$. The map $f : GS \to G/H$ defined by $f(gs) = gH$ is a definable G map and $f^{-1}(eH) = S$. □

Lemma 4.6. *Let G be a compact definable group and X a definable G set. If $X - X^G$ is definably G imbeddable into some orthogonal G reprepsentation space, then so does X.*

Proof. Let X/G is a definable subset of $\mathbb{R}^k$ and let $\pi : X \to X/G \subset \mathbb{R}^k$ be the orbit map. Then the map $h : X/G \to \mathbb{R}$ defined by $h(x) = \inf\{||x - y|| | y \in X^G/G\}$ is a definable map, where $||z||$ denotes the standard norm of z. Moreover $\tilde{h} : X \to \mathbb{R}, \tilde{h} = h \circ \pi$ is a G invariant definable map.

Let $f : X - X^G \to \Omega$ be a definable G imbedding. By replacing Ω by $\Omega \oplus \mathbb{R}$, we may assume that $||f(x)|| = 1$ for all $x \in X - X^G$, where $\mathbb{R}$ denotes the one dimensional trivial real representation space of G.

The map $\tilde{f} : X \to \Omega$ defifned by $\tilde{f}(x) = \begin{cases} h(x)f(x), & x \in X - X^G \\ 0, & x \in X^G \end{cases}$ is a continuous G map (see P22 [12]). By construction, $\tilde{f}$ is definable.

Then the map $F : X \to \mathbb{R}^k \oplus \Omega$ defined by $F(x) = (\pi(x), \tilde{f}(x))$ is a definable G map, where $\mathbb{R}^k$ denotes the k-dimensional trivial real G representation space. By construction, F is a definable G imbedding. □

The following is a definable partition of unity.

Proposition 4.7 (e.g. 6.3.7 [2]). *Let X be a definable subset of $\mathbb{R}^n$ and $\{U_i\}_{i=1}^l$ a finite definable open covering of X. Then there exist definable functions $\lambda_1, \dots, \lambda_l : X \to \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

The following is the equivariant version of Proposition 4.7.

Proposition 4.8. *Let G be a compact definable group, X a definable G set and $\{U_i\}_{i=1}^l$ a finite open covering of X by G invariant definable sets. Then there exist G invariant definable functions $\lambda_1, \dots, \lambda_l : X \to \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

Proof. Let $\pi : X \to X/G$ be the orbit map. Since π is a definable open map, $\{\pi(U_i)\}_{i=1}^l$ a finite definable open covering of X/G. By Proposition 4.7, there exist definable functions $\lambda'_1, \dots, \lambda'_l : X/G \to \mathbb{R}$ such that $0 \leq \lambda'_i \leq 1$, supp $\lambda'_i \subset \pi(U_i)$ and $\sum_{i=1}^l \lambda'_i(x) = 1$ for any $x \in X/G$. Thus $\lambda_1 = \lambda'_1 \circ \pi, \dots, \lambda_l = \lambda'_l$ are the required G invariant definable functions. □

Proposition 4.9. *Let G be a compact definable group and X a definable G set. If*

$\{U_i\}_{i=1}^k$ is a finite open covering of X by G invariant definable sets and each U_i is definably G imbeddable into a definable orthogonal G representation space Ω_i, then X is definably G imbeddable into a definable orthogonal G representation space Ω.

Proof. By Proposition 4.8, there exist G invariant definable functions $\lambda_1, \dots, \lambda_k : X \to [0,1]$ such that $0 \leq \lambda_i \leq 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.

Let $\phi_i : U_i \to \Omega_i$ be a definable G imbedding. Then the map $\psi_i : X \to \Omega_i$ defined by $\psi_i(x) = \begin{cases} \lambda_i(x)\phi_i(x), & x \in U_i \\ 0, & x \in X - U_i \end{cases}$ is a definable G map. Let $\mathbb{R}^k$ denote the k-dimensional trivial real G representation space. Then the map $\phi : X \to \mathbb{R}^k \oplus \Omega_1 \oplus \cdots \oplus \Omega_k, \phi(x) = (\lambda_1(x), \dots, \lambda_k(x), \psi_1(x), \dots, \psi_k(x))$ is the required definable G imbedding. □

Proof of Theorem 1.2. We proceed by induction and we assume that the theorem is true for all proper definable (closed) subgroups of G. By Lemma 4.6, it is enough to prove that $X - X^G$ is definably G imbeddable into a definable orthogonal G representation space. By Theorem 1.1, there exist a finite number of definable H_i slices $S_1, \dots, S_k$ of $X - X^G$ such that $GS_1, \dots, GS_k$ cover $X - X^G$. Applying the inductive hypothesis to H_i, there exist a definable orthogonal H_i representation space Ω_i and a definable H_i imbedding $\phi_i : S_i \to \Omega_i$. By Proposition 4.5, there exists a definable H_i imbedding ψ_i of Ω_i onto a definable H_i kernel in some definable orthogonal G representation space Ξ_i. Then the map $f_i : GS_i \to \Xi_i$ defined by $f_i(gs) = g\psi_i(\phi_i(s))$ is a definable G imbedding. Since $\{GS_i\}_{i=1}^k$ is a finite open covering of $X - X^G$ by G invariant definable sets and Proposition 4.9, $X - X^G$ admits a definable G imbedding into a definable orthogonal G representation space. □

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