Definable proper actions and equivariant definable Tietze extension

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Abstract

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field R. Let G be a definable group and X a definable proper definable G set. We prove that X has only finitely many orbit types. We also prove equivariant definable Tietze extension theorem.

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1 . Introduction.

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field R.

General references on o-minimal structures are [2], [3], and also see [9]. Any definable category is a generalization of the semialgebraic category and the definable category on $\mathcal{R} = (R, +, \cdot, <)$ coincides with the semialgebraic one. It is known in [8] that there exist uncountably many o-minimal expansions of the field $\mathbb{R}$ of real numbers.

Let G be a definable group. A *definable G set* means a pair consisting of a definable set X and a group action $\phi : G \times X \to X$ such that ϕ is definable. A definable map between definable sets is called *definably proper* if the inverse image of every definably compact definable set is definably compact. We call a definable G set X a *proper definable G set* if the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (gx, x)$ is definably proper.

Let G be a definable group. We can define *orbit types* as well as G is definably compact ([5]).

Theorem 1.1. *Let G be a definable group. Then every proper definable G set X has only finitely many orbit types.*

Theorem 1.1 is proved the case where R is the field $\mathbb{R}$ of real numbers ([5]).

The following theorem is an equivariant version of definable Tietze extension theorem [1]

Theorem 1.2. *Let G be a definably compact definable group, X a definable G set and A a G invariant definably compact definable subset of X. Every G invariant definable function $f : A \to R$ is extensible to a G invariant definable function $F : X \to R$ with $F|A = f$.*

2 . Proof of results.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset R^n \times R^m$) is a definable set. A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A definable subset X of R^n is *definably compact* if for every definable map $f : (a,b)_R \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in X, where $(a,b)_R = \{x \in R | a \le x < b\}, -\infty \le a < b \le \infty$. A definable subset X of R^n is definably compact if and only if X is closed and bounded ([7]). Note that if X is a definably compact definable set and $f : X \to Y$ is a definable map, then $f(X)$ is definably compact.

We say that two homogeneous proper definable G sets are *equivalent* if they are definably G homeomorphic. Let (G/H) denote the equivalence class of G/H. The set of all equivalence classes of homogeneous proper definable G sets has a natural order defined as $(X) \ge (Y)$ if there exists a definable G map $X \to Y$. By the definition the reflexivity and the transitivity clearly hold If $(X) = (G/H)$ and $(Y) = (G/K)$, then $(X) \ge (Y)$ if and only if H is conjugate to a definable subgroup of K. By a way similar to the proof of 4.1 [5], we have the following lemma.

Lemma 2.1. *Let G be a definable group, H a definable subgroup of G and $g \in G$. If $gHg^{-1} \subset H$, then $gHg^{-1} = H$.*

By Lemma 2.1, the anti-symmetry is true.

By a way to similar to the proof of 1.1 [5], we have Theorem 1.1.

Note that every definable subgroup of a definable group is closed ([6]) and a closed subgroup of a definable group is not necessarily definable. For example $\mathbb{Z}$ is a closed subgroup of $\mathbb{R}$ but not a definable subgroup of $\mathbb{R}$.

Recall existence of definable quotient.

Theorem 2.2 (Existence of definable quotient, 10.2.18 [2])**.** *Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set, and the orbit map $\pi : X \to X/G$ is definable, surjective and definably proper.*

The following theorem is the topological case of Tietze extension theorem.

Theorem 2.3 (Tietze extension theorem)**.** *Let X be a normal space and A a closed subset of X. Then every continuous map $f : A \to \mathbb{R}$ is extensible to a continuous map $F : X \to \mathbb{R}$ with $F|A = f$.*

The following theorem is the definable case of Tietze extension theorem.

Theorem 2.4 (Definable Tietze extension theorem [1])**.** *Let A be a definable closed subset of R^n. Then every definable map $f : A \to R$ is extensible to a definable map $F : R^n \to R$ with $F|A = f$.*

A definable map $f : X \to Y$ is *definably closed* if for any definable closed subset A of X, $f(A)$ is a definable closed subset of Y.

Theorem 2.5 ([4])**.** *Let $f : X \to Y$ be a definable map. Then f is definably proper if and only if f is definably closed and has definably compact fibers.*

Proof of Theorem 1.2. By Theorem 2.2, X/G exists as a definable set in R^n and the projection $\pi : X \to X/G$ is a surjective definable definably proper map. By Theorem 2.5 and A is definably compact, $\pi(A)$ is a definable closed subset of R^n. Since f is a G invariant definable map, it induces a definable map $f' : f(A) \to R$ with $f = \pi \circ f'$. By Theorem 2.4, there exists a definable map $F : R^n \to R$ with $F|f(A) = f'$. Hence $H = \pi \circ F$ is the required map. $\square$

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