Definable isotopies

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Abstract

Let X be a definable C^r manifold, Y_1, Y_2 definably compact definable C^r submanifolds of X such that $\dim Y_1 + \dim Y_2 < \dim X$ and Y_1 has a trivial normal bundle. We prove that there exists a definable isotopy $\{h_p\}_{p \in J}$ such that $h_0 = id_X$ and $h_1(Y_1) \cap Y_2 = \emptyset$.

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1 . Introduction.

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Everything is considered in $\mathcal{N}$, the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{N}$", each definable map is assumed to be continuous and $2 \le r < \infty$.

General references on o-minimal structures are [2], [3], also see [10].

In this paper we consider definable isotopies of definable C^r manifolds and gradient like vector fields of definable C^r Morse functions when $2 \le r < \infty$. Definable C^r Morse functions in an o-minimal expansion of the standard structure of a real closed field are considered in [9].

Definable C^r manifolds are studied in [9], [1], and definable $C^r G$ manifolds are studied in [4]. If R is the field $\mathbb{R}$ of real numbers, then definable $C^r G$ manifolds are considered in [8], [7], [6] [5].

Theorem 1.1 (10.7 [1]). *Every definably compact definable C^r manifold X is definably C^r diffeomorphic to a definable C^r submanifold of some R^n.*

By Theorem 1.1, we may assume that a definably compact definable C^r manifold X is a definable C^r submanifold of some R^n.

Let X be a definable C^r manifold and J an open interval including $[0, 1]_R = \{x \in R | 0 \le x \le 1\}$. A family $\{h_t\}_{t \in J}$ of definable C^r diffeomorphisms of X is a *definable isotopy* of X if h_t is identity if $t \le 0$, $h_t = h_1$ is a definable C^r diffeomorphism if $t \ge 1$ and $H : X \times J \to X \times J, H(x, t) = (h_t(x), t)$ is a definable C^r diffeomorphism.

Theorem 1.2. *Let X be a definable C^r manifold, Y_1, Y_2 definably compact definable C^r submanifolds of X such that $\dim Y_1 + \dim Y_2 < \dim X$ and Y_1 has a trivial normal bundle. Then there exists a definable isotopy $\{h_p\}_{p \in J}$ such that $h_0 = id_X$ and $h_1(Y_1) \cap Y_2 = \emptyset$.*

Let X be a definable C^r manifold. Then as in the standard version, we can define the tangent bundle TX of X. A *definable* C^{r-1}

vector field is a definable C^{r-1} section of TX.

Definition 1.3. *Let X be a definable C^r manifold and $f : X \to R$ a definable Morse function. A definable C^{r-1} vector field Ξ on X is a gradient like vector field of f if the following two conditions are satisfied.*

(1) $(X \cdot f)(p) > 0$ if p is not a critical point of f.

(2) If p is a critical point of f with index λ, then there exists a definable coordinate neighborhood $(x_1, \dots, x_n)$ such that $f = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ and Ξ is a gradient vector field of f.

Theorem 1.4. *Let X be a definably compact definable C^r manifold and $f : X \to R$ a definable Morse function. Then there exists a gradient like vector field of f.*

2 . Preliminaries.

Let $W_1 \subset R^n, W_2 \subset R^m$ be definable open sets and $f : W_1 \to W_2$ a definable map. We say that f is a *definable C^r map* if f is of class C^r. A definable C^r map is a *definable C^r diffeomorphism* if f is a C^r diffeomorphism.

Definition 2.1. *A Hausdorff space X is an n-dimensional definable C^r manifold if there exist a finite open cover $\{U_i\}_{i=1}^k$ of X, finite open sets $\{V_i\}_{i=1}^k$ of R^n, and a finite collection of homeomorphisms $\{\phi_i : U_i \to V_i\}_{i=1}^k$ such that for any i, j with $U_i \cap U_j \neq \emptyset$, $\phi_i(U_i \cap U_j)$ is definable and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. This pair $(\{U_i\}_{i=1}^k, \{\phi_i : U_i \to V_i\}_{i=1}^k)$ of sets and homeomorphisms is called a definable C^r coordinate system.*

A definable C^r manifold X is *definably compact* if for every $a, b \in R \cup \{\infty\} \cup \{-\infty\}$ with $a < b$ and for every definable map $f : (a, b) \to X$, $\lim_{x \to a+0} f(x)$ and $\lim_{x \to b-0} f(x)$ exist in X.

If $R = \mathbb{R}$, then for any definable C^r manifold X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1\}$ is definably compact but not compact.

Let X be an m-dimensional definable C^r manifold and $f : X \to R$ a definable C^r function. A point $p \in X$ is a *critical point* of f if the differential of f at p is zero. If p is a critical point of f, then $f(p)$ is called a *critical value* of f. Let p be a critical point of f and $(U, \phi : (U, p) \to (V, 0))$ a definable C^r neighborhood around p. The critical point p is *nondegenerate* if the Hessian of $f \circ \phi^{-1}$ at 0 is nonsingular. Direct computations show that the notion of nondegeniricity does not depend on the choice of a local coordinate neighborhood. We say that f is a *definable Morse function* if every critical point of f is nondegenerate.

3 Proof of our results

The following result is a definable version of Sard's Theorem.

Theorem 3.1 (3.5 [1]). *Let $X_1 \subset R^s$ and $X_2 \subset R^t$ be definable C^r manifolds of dimension m and n, respectively. Let $f : X_1 \to X_2$ be a definable C^r map. Then the set of critical values of f has dimension less than n.*

To prove Theorem 1.2, we have the following lemma.

Lemma 3.2. *Let D^k be the k-dimensional closed unit disk of R^k and $0 < a < 1$. Then there exists a definable isotopy $\{h_t\}_{t \in J}$ such that $h_0 = id$ and $h_1(0, \dots, 0, 0) = (0, \dots, 0, a)$.*

Proof. Take a definable C^r function $f : R \to R, f(x) = \begin{cases} 1, |x| < \frac{1}{3} \\ 0, |x| > \frac{1}{2} \end{cases}$.

If $\epsilon > 0$ is sufficiently small, then $f_\epsilon(x) = \epsilon f(x) + x$ is increasing, $f(x) = x$ if $|x| > \frac{1}{2}$ and $f_\epsilon(0) = \epsilon$.

Take a definable C^r function $\rho_\epsilon : R \to R, \rho(x) = \begin{cases} 0, x < \frac{\epsilon}{2} \\ 1, x > \epsilon \end{cases}$.

We define $g_\epsilon : R^k \to R, g_\epsilon(x_1, \dots, x_k) = (1 - \rho_\epsilon(x_1^2 + \dots x_{k-1}^2)) f_\epsilon(x_k) + \rho_\epsilon(x_1^2 + \dots x_{k-1}^2) x_k$.

Then $g_\epsilon(x_1,\dots,x_k) = f_\epsilon(x_k)$ if $x_1^2+\cdots+x_{k-1}^2 < \frac{\epsilon}{2}$ and $g_\epsilon(x_1,\dots,x_k) = x_k$ if $x_1^2+\cdots+x_{k-1}^2 > \epsilon$. Moreover $g_\epsilon(x_1,\dots,x_k) = x_k$ if $|x_k| > \frac{1}{2}$, $g_\epsilon(x_1,\dots,x_k)$ is increasing with respect to x_k and $g_\epsilon(0,\dots,0) = f_\epsilon(0) = \epsilon$. Then the map $h : D^k \to D^k$ defined by $h(x_1,\dots,x_{k-1},x_k) = (x_1,\dots,x_{k-1},g_\epsilon(x_1, \dots,x_k))$ is the identity on a definable open neighborhood of ∂D^k, $h(0,\dots,0,0) = (0, \dots,0,\epsilon)$ and h is a definable C^r diffeomorphism. We define a definable isotopy $\{h_t\}$ of D^k by $h_t(x_1,\dots,x_{k-1},x_k) = (x_1,\dots,x_{k-1}, \rho_\epsilon g_\epsilon(x_1,\dots,x_{k-1},x_k) + (1-\rho_\epsilon(t)x_k)$. Then $h_t = id$ if $t \le 0$, $h_t = h$ if $t \ge \epsilon$ and $h_1(0,\dots,0,0) = (0\dots 0,\epsilon)$.

Let $\epsilon < a < 1$. We now construct a definable isotopy $\{H_t\}$ of D^k such that $H_1(0,\dots, 0,0) = (0,\dots,0,a)$. For a sufficiently small $\delta > 0$, take a definable C^r function $\sigma : R \to R, \sigma(x) = \begin{cases} \frac{\epsilon}{a}, x < a+\delta \\ 1, x > a+2\delta \end{cases}$.

Then the map $H : D^k \to D^k$ defined by $H(x_1,\dots,x_k) = (\sigma(||x||)x_1,\dots,\sigma(||x||)x_k)$ is a definable C^r diffeomorphism, where $||x||$ denotes the standard norm of R^k. H is the identity on a definable open neighborhood of ∂D^k and $H(0,\dots,0,a) = (0,\dots,0,\epsilon)$.

Thus $\{H^{-1}\circ h_t \circ H\}_{t\in J}$ is a definable isotopy such that the identity if $t \le 0$, $H^{-1}\circ h_1 \circ H(0,\dots,0,0) = (0,\dots 0,a)$. $\square$

Theorem 3.3. *Let D^k be the k-dimensional closed unit disk of R^k and $p,q \in IntD^k$. Then there exists a definable isotopy $\{h_t\}_{t\in J}$ such that $h_0 = id$, $h_1(p) = q$ and h_t is identity on a definable open neighborhood of ∂D^k.*

Proof. We prove that the theorem the case where $p = 0$ and $q \ne 0$. Since $q \ne 0$, $a = ||q||$ satisfies $0 < a < 1$. Let $(b_1,\dots,b_k) = \frac{1}{a}(q_1,\dots,q_k)$, where $q = (q_1,\dots,q_k)$. Since $||(b_1,\dots,b_k)|| = 1$, we can take an orthogonal matrix B including $[b_1,\dots,b_k]$ as the n-th column. Hence $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = B \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

Therefore $\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = B \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a \end{bmatrix}$.

By Lemma 3.2 and composing the matrix operation of B, we have a definable isotopy of D^k such that $h_1(0) = q$.

By the above argument, we have a definable isotopy of D^k such that $h_1(p) = 0$. Composing these two definable isotopies, we have the required definable isotopy. $\square$

Remark 3.4. *(1) Theorem 3.3 is a definable version of the classical result.*

(2) If $\mathcal{N} = (\mathbb{R},+,\cdot,<,exp,\dots)$, then we can take $r = \infty$.

Proof of Theorem 1.2. By assumption, S_1 has a definable open neighborhood U which is definably C^r diffeomorphic to $S_1 \times int(D^{k-s_1})$. We identify U with $S_1 \times int(D^{k-s_1})$. Let $\pi : S_1 \times int(D^{k-s_1})$ be the projection onto the second factor. By assumption, $\dim(S_2 \cap U) = s_2 < k-s_1 = \dim D^{k-s_1}$. Hence $\dim \pi(S_2\cap U) < \dim\ int(D^{k-s_1})$. By Theorem 3.1, there exists p_0 near 0 such that $p_0 \notin \pi(S_2 \cap U)$. By Lemma 3.2, there exists a definable isotopy $\{j_t\}_{t\in J}$ of $int(D^{k-s_1})$ such that

(1) $j_0 = id$ and $j_1(0) = p_0$.

(2) For any t, j_t is the identity outside of $\frac{1}{2}D^{k-s_1}$.

The family $\{H_t\}_{t\in J}$ defined by $h_t(p,x) = (p,j_t(x)), \forall (p,x) \in S_1 \times int(D^{k-s_1})$ is a definable isotopy of U. Since this is the identity outside of $S_1 \times \frac{1}{2}D^{k-s_1}$, we can extend it to a definable isotopy $\{h_t\}_{t\in J}$ of X. By construction, $h_1(S_1) = S_1 \times \{p_0\}$ in U. Since the choice of p_0, $(S_1\times\{p_0\}) \cap (S_2\cap U) = \emptyset$. Therefore $h_1(S_1)\cap S_2 = \emptyset$. $\square$

Theorem 3.5 (5.8 [1])**.** *Let $X \subset R^l$ be a definable C^r manifold. Given two disjoint definable sets $F_0, F_1 \subset X$ closed in X, there exists a definable C^p function $\delta : X \to R$ which is 0 exactly on F_0, 1 exactly on F_2 and $0 \le \delta \le 1$.*

Lemma 3.6 (6.3.6 [2])**.** *Let $A \subset R^n$ be a definable set which is the union of definable* *open subsets $U_1, \ldots, U_n$ of A. Then A is the union of definable open subsets $W_1, \ldots, W_n$ of A with $cl_A(W_i) \subset U_i$ for $i = 1, \ldots, n$, where $cl_A(W_i)$ denotes the closure of W_i in A.*

The following is the Morse's lemma in the definable category.

Lemma 3.7 (A7 [9]). *Let $r \geq 0$, X a definable C^{r+2} manifold of dimension n, $f : X \to R$ a definable C^{r+2} function and $p \in X$ a nondegenerate critical point of f. Then there exists a definable C^r coordinate system $(U, \phi)_2$ of X at p_2 such that $f = -y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1} \cdots + y_n$.*

Proof of Theorem 1.4. By the definition of definable C^r manifolds, there exists a finite number of definable coordinate system $\{U_i\}_{i=1}^k$ of X. By Lemma 3.6 and since X is definably compact, replacing $\{U_i\}_{i=1}^k$, if necessary, there exists finite number of definably compact sets $\{K_i\}_{i=1}^k$ such that $K_i \subset U_i$ and $\cup_{i=1}^k K_i = X$. Moreover we may assume that for any critical point p_0, p_0 lies in a unique U_i and U_i satisfies Lemma 3.7.

For any i, we define the gradient vector field X_f of f in U_i by

$X_f = \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_m}\frac{\partial}{\partial x_m}$. Then for any non-critical point, $X_f \cdot f > 0$. By Theorem 3.5, there exists a definable C^r function $h_i : U_i \to R$ such that $0 \leq h \leq 1$, $h_i = 1$ on a definable open neighborhood V_i of K_i and $h_i = 0$ outside a definably compact set L_i containing V_i with $L_i \subset U_i$. Each h_i is extensible to X defining 0 outside of U_i. Then we have a definable C^{r-1} vector field $\Xi = \sum_{i=1}^k h_i X_i$ of X.

We now prove Ξ is the required vector field. Let p be a non-critical point. Then $(X_i \cdot f)(p) > 0$ if $p \in U_i$ and $(h_i X_f \cdot f)(p) \geq 0$ otherwise. Since $X = \cup_{i=1}^k K_i$, there exists a K_i such that $p \in K_i$. Since $h_i = 1$ on K_i, $(X_f \cdot f)(p) > 0$. Thus $X \cdot f > 0$.

Let p be a critical point. Then there exist a sufficiently small definable open neighborhood V of p contained in a unique U_i. Since $h_i = 1$ on V and f is written in the standard form, $h_i X_i$ is a form in the Definition 1.3 (2). Since any other $h_i X_i$ is 0 on V, X is a form in the Definition 1.3 (2). □

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