Definable G vector bundles over a definable G set with free action

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Abstract

We prove that the set of isomorphism classes of definable G vector bundles over a definable G set X is in one–to-one correspondence to that of definable vector bundles over a definable set X/G when the action on X is free.

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1 . Introduction.

Let G be a compact Lie group. It is well-known that the set of isomorphism classes of G vector bundle over a G space space with free action corresponds bijectively to the set of isomorphism classes of vector bundles over the orbit space [1].

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Everything is considered in $\mathcal{N}$ and the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{N}$", each definable map is assumed to be continuous.

General references on o-minimal structures are [2], [3], also see [9].

In this paper we prove that the set of isomorphism classes of definable G vector bundles over a definable G set X is in one–to-one correspondence to that of definable vector bundles over a definable set X/G when the action on X is free.

Theorem 1.1. *Let G be a definably compact definable group and X a definable G set. If G acts on X freely, then the set of isomorphism classes of definable G vector bundles over X corresponds bijectively to the set of isomorphism classes of definable vector bundles over X/G.*

If R is the field $\mathbb{R}$ of real numbers, then a semialgebraic case of Theorem 1.1 is proved in [7]. Definable G vector bundles are studied in [6], [5], [4].

2 . Proof of our result.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset R^n \times R^m$) is a definable set. A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A definable subset X of R^n

is *definably compact* if for every definable map $f : (a,b)_R \to X$, there exist the limits $\lim_{x\to a+0} f(x), \lim_{x\to b-0} f(x)$ in X, where $(a,b)_R = \{x \in R | a < x < b\}, -\infty \le a < b \le \infty$. A definable subset X of R^n is definably compact if and only if X is closed and bounded ([8]). Note that if X is a definably compact definable set and $f : X \to Y$ is a definable map, then $f(X)$ is definably compact.

If R is the field of real numbers $\mathbb{R}$, then for any definable subset X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general, a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \le x \le 1\}$ is definably compact but not compact.

Let G be a definably compact definable group. A group homomorphism from G to some $O_n(R)$ is a *representation* if it is definable, where $O_n(R)$ means the nth orthogonal group of R. A *representation space* of G is R^n with the orthogonal action induced from a representation of G. We say that a G invariant definable subset of a representation space of G is a *definable G set*.

Recall existence of definable quotient.

Theorem 2.1. *(Existence of definable quotient (e.g. 10. 2.18 [2])). Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.*

Let X, Y be a definable G sets. A definable map $f : X \to Y$ is a *definable G map* if for any $g \in G, x \in X, f(gx) = gf(x)$.

Let G be a definably compact definable group and $\eta = (E, \pi, X)$ a definable G vector bundle. By Theorem 2.1, E/G and X/G are definable sets, and π induces a definable map $\pi/G : E/G \to X/G$.

Proposition 2.2. *Let G be a definably compact definable group and $\eta = (E, \pi, X)$ a definable G vector bundle. Then the quotient bundle $\eta = (E/G, \pi/G, X/G)$ is a definable vector bundle.*

To prove Proposition 2.2, we need two lemmas.

Lemma 2.3. *Let $X \subset R^n, Y \subset R^m$ be definable sets. If $f : X \to Y$ is a surjective definable map, then there exists a definable map $h : Y \to X$ such that $f \circ h = id_Y$.*

Proof. Replacing X by the graph of f, we may assume that f is a restriction of the projection $R^n \times R^m \to R^m$. We now prove the case where $n = 1$. By cell decomposition theorem (e.g. [2]), there exists a cell decomposition compatible with X. By definition of cells, we can construct a definable map h satisfying $f \circ h = id_Y$. In general case, we have the required map because the projection is the composition of $R^n \times R^m \to R^{n-1} \times R^m \to \cdots \to R^m$. □

Lemma 2.4. *Let X, Y, Z be definable sets, $s : X \to Y, t : Y \to Z$ definable maps and $u : Y \to Z$ a map. If t is surjective and $s = u \circ t$, then u is a definable map.*

Proof. By Lemma 2.3, there exists a definable map $q : Y \to X$ such that $t \circ q = id_Y$. Therefore $u = u \circ id_Y = u \circ t \circ q = s \circ q$. □

Proof of Theorem 1.1. Let $\eta = (E, \pi, X)$ be a definable G vector bundle over X. Let q_X, q_E be the orbit maps of $X \to X/G, E \to E/G$, respectively. Set $p = q_X \circ \pi$. Then $p = \pi/G \circ q_E$. By Lemma 3.4, $\pi/G : E/G \to X/G$ is a definable map.

Let $\{U_1, \pi_1\}$ be a definable trivialization of η. Since any fiber of η has the trivial action, we have a definable trivialization $\{q_X, \pi'_1\}$ is of η/G, where π'_1 is the map satisfying $(p_X | U_i \times id) \circ \pi_i = \pi'_1 \circ q_E | \pi^{-1}(U_i)$. Hence $\eta/G = (E/G, \pi/G, X/G)$ is a definable vector bundle over X/G. Thus we can define the map $F : VEC(X) \to VEC(X/G)$ by $F(\eta) = \eta/G$, where $VEC(X)$ (resp. $VEC(X/G)$) denotes the set of isomorphism classes of definable G vector bundles (resp. the set of isomorphism classes of definable vector bundles) over X (resp. X/G). The map $K : VEC(X/G) \to VEC(X)$ defined by $K(\xi) = q_X^*(\xi)$ satisfies $F \circ K = id, K \circ F = id$. Thus the proof is complete. □

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