# Definable $C^r$ fiber bundle structures of a definable fiber bundle

Tomohiro Kawakami
Department of Mathematics, Faculty of Education, Wakayama University,
Sakaedani, Wakayama 640-8510, Japan
kawa@center.wakayama-u.ac.jp

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## Abstract

Let $G$ and $K$ be definably compact subgroups of orthogonal groups and $0 \leqq r < \infty$. We prove that every definable fiber bundle over an affine definable $C^r$ manifold whose structure group is $K$ admits a unique strongly definable $C^r$ fiber bundle structure up to definable $C^r$ fiber bundle isomorphism.

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## 1 . Introduction.

J. Bochnak, M. Coste and M.F. Roy proved that the following theorem.

**Theorem 1.1** (12.7.14. [1]). *Every semialgebraic vector bundle over an affine Nash manifold admits a unique strongly Nash vector bundle structure up to Nash vector bundle isomorphism.*

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (R, +, \cdot, <)$ of a real closed field.

Everything is considered in $\mathcal{N}$ and the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{N}$", each definable map is assumed to be continuous.

General references on o-minimal structures are [3], [4], also see [17].

In this paper we prove the definable fiber bundle version of the above result.

**Theorem 1.2.** *Let $\eta = (E, p, X, F, K)$ be a strongly definable fiber bundle over an affine definable $C^r$ manifold and $K$ an affine definably compact definable $C^r$ group.*

*(1) There exists a strongly definable $C^r$ fiber bundle $\zeta$ over $X$ such that $\zeta$ is fiber bundle isomorphic to $\eta$.*

*(2) If $\zeta'$ is another strongly definable $C^r$ fiber bundle over $X$ such that $\zeta'$ is fiber bundle isomorphic to $\eta$, then $\zeta'$ and $\zeta$ are definably $C^r$ fiber bundle isomorphic.*

*In particular, (1) and (2) say that $\eta$ admits a unique definable $C^r$ fiber bundle structure up to definable $C^r$ fiber bundle isomorphism.* □

If $R = \mathbb{R}$, then Theorem 1.1 is proved in [7].

**Theorem 1.3.** *Let $G$ be a finite grouop and $0 \leqq r < \infty$. Every definable $G$ vector bundle over an affine definable $C^rG$ manifold admits a unique strongly definable $C^rG$ vector bundle structure up to definable $C^rG$ vector bundle isomorphism.*

# 2 . Proof of our results.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of $f$ $(\subset X \times Y \subset R^n \times R^m)$ is a definable set. A group $G$ is a *definable group* if $G$ is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A definable subset $X$ of $R^n$ is *definably compact* if for every definable map $f : (a,b)_R \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in $X$, where $(a,b)_R = \{x \in R | a \leq x < b\}, -\infty \leq a < b \leq \infty$. A definable subset $X$ of $R^n$ is definably compact if and only if $X$ is closed and bounded ([16]). Note that if $X$ is a definably compact definable set and $f : X \to Y$ is a definable map, then $f(X)$ is definably compact.

If $R$ is the field of real numbers $\mathbb{R}$, then for any definable subset $X$ of $\mathbb{R}^n$, $X$ is compact if and only if it is definably compact. In general, a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1\}$ is definably compact but not compact.

A *definable space* is an object obtained by pasting finitely many definable sets together along definable open subsets, and definable maps between definable spaces are defined similarly (see Chapter 10 [3]). Definable spaces are generalizations of semialgebraic spaces in the sense of [2].

Recall the definition of definable fiber bundles [12].

**Definition 2.1.** (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *definable fiber bundle* over $X$ with fiber $F$ and structure group $K$ if the following two conditions are satisfied:

(a) The total space $E$ is a definable space, the base space $X$ is a definable set, the structure group $K$ is a definable group, the fiber $F$ is a definable set with an effective definable $K$ action, and the projection $p : E \to X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of $\eta$ such that each $U_i$ is a definable open subset of $X$, $\{U_i\}_i$ is a finite open covering of $X$. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where $\pi_i$ stands for the projection $U_i \times F \to F$. For any $i$ and $j$ with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations *definable*.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\overline{f} : E \to E'$ is said to be a *definable fiber bundle morphism* if the following two conditions are satisfied:

(a) The map $\overline{f}$ covers a definable map, namely there exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

(b) For any $i, j$ such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in $K$, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

We say that a bijective definable fiber bundle morphism $\overline{f} : E \to E'$ is a *definable fiber bundle equivalence* if it covers a definable homeomorphism $f : X \to X'$ and $(\overline{f})^{-1} : E' \to E$ is a definable fiber bundle morphism covering $f^{-1} : X' \to X$. A definable fiber bundle equivalence $\overline{f} : E \to E'$ is called a *definable fiber bundle isomorphism* if $X = X'$ and $f = id_X$.

(3) A continuous section $s : X \to E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable section* if for any $i$, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

(4) We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal definable fiber bundle* if $F = K$ and the $K$ action on $F$ is defined by the multiplication of $K$. We write $(E, p, X, K)$ for $(E, p, X, F, K)$.

A *definable $C^r$ manifold* is a Hausdorff space with a finite system of charts whose transition functions are definable, and definable $C^r$ maps, definable $C^r$ diffeomorphisms and definable $C^r$ imbeddings are defined similarly ([13], [10]). A definable $C^r$ manifold is *affine* if it is definably $C^r$ imbeddable into some $R^n$. If $\mathcal{M} = \mathbb{R}$, a definable $C^\omega$ manifold (resp. affine definable $C^\omega$ manifold) is called a *Nash manifold* (resp. an *affine Nash manifold*). By [11], if $R = \mathbb{R}$ and $0 \leq r < \infty$, then every definable $C^r$ manifold is affine. The definable $C^\omega$ case is complicated. Even if $\mathcal{M} = \mathcal{R}$, it is known that for every compact or compactifiable $C^\omega$ manifold of positive dimension admits a continuum number of distinct nonaffine Nash manifold structures (IV.1.3 [18]), and its equivariant version is proved in [14].

A *definable $C^r G$ action* on a definable $C^r$ manifold $X$ is a group action $G \times X \to X$ such that it is a definable $C^r$ map.

Recall the definition of definable $C^r$ fiber bundles [10].

**Definition 2.2** ([10]). (1) A definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable $C^r$ fiber bundle* if the total space $E$ and the base space $X$ are definable $C^r$ manifolds, the structure group $K$ is a definable $C^r$ group, the fiber $F$ is a definable $C^r K$ manifold with an effective action, the projection $p$ is a definable $C^r$ map and all transition functions of $\eta$ are definable $C^r$ maps. A *principal definable $C^r$ fiber bundle* is defined similarly.

(2) *Definable $C^r$ fiber bundle morphisms*, *definable $C^r$ fiber bundle equivalences*, *definable $C^r$ fiber bundle isomorphisms* between definable $C^r$ fiber bundles and *definable $C^r$ sections* of a definable $C^r$ fiber bundle are defined similarly.

Recall existence of definable quotient.

**Theorem 2.3.** *(Existence of definable quotient (e.g. 10. 2.18 [3])). Let $G$ be a definably compact definable group and $X$ a definable $G$ set. Then the orbit space $X/G$ exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.*

By a similar proof of 2.10 [15] and Theorem 2.3, we have the following.

**Proposition 2.4.** *Let $(E, p, X, K)$ be a principal definable fiber bundle, $F$ a definable set with an effective definable $K$ action and $K$ a definably compact definable group. Then $(E \times_K F, p', X, F, K)$ is a definable fiber bundle, where $p' : E \times_K F \to X$ denotes the projection defined by $p'([z, f]) = p(z)$.*

We have the definable $C^r$ version of Proposition 2.4 similarly.

**Proposition 2.5.** *Let $(E, p, X, K)$ be a principal definable $C^r$ fiber bundle over a definable $C^r$ manifold $X$, $F$ an affine definable $C^r$ manifold with an effective definable $C^r K$ action and $K$ an affine definably compact definable $C^r$ group. Then $(E \times_K F, p', X, F, K)$ is a definable $C^r$ fiber bundle, where $p' : E \times_K F \to X$ denotes the projection defined by $p'([z, f]) = p(z)$.*

As a corollary of Proposition 2.5, we have the following proposition.

**Proposition 2.6.** *Let $\mathcal{B}_K = (B_K, p_K, X_K)$ be the $n$-universal principal bundle relative to $K$, $F$ an affine definable $C^r$ manifold with an effective definable $C^r K$ action and $K$ an affine definably compact definable $C^r$ group. Then the associated fiber bundle $\mathcal{B}_K[F] := (E, p, X_K, F, K)$ is a definable $C^r$ fiber bundle.*

**Theorem 2.7** ([5])**.** *Let $X \subset R^n, Y \subset R^m$ be definable $C^r$ manifolds and $0 \leqq s < r < \infty$. Every definable $C^s$ map $f : X \to Y$ is approximated by a definable $C^r$ map $h : X \to Y$ in the definable $C^s$ topology.*

**Definition 2.8.** *(1) A definable fiber bundle $\eta = (E, p, X, F, K)$ is strongly $definable$ if there exist the $n$-universal bundle $\mathcal{B}_K$ and a definable map $f : X \to X_K$ such that $f^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to $\eta$.*

*(1) A definable $C^r$ fiber bundle $\eta = (E, p, X, F, K)$ is strongly $definable$ if there exist the $n$-universal bundle $\mathcal{B}_K$ and a definable $C^r$ map $f : X \to X_K$ such that $f^*(\mathcal{B}_K[F])$ is definably $C^r$ fiber bundle isomorphic to $\eta$.*

*Proof of Theorem* 1.2. (1) Since $\eta$ is strongly definable, there exists the $n$-universal bundle $\mathcal{B}_K$ and a definable map $f : X \to X_K$ such that $f^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to $\eta$. By Theorem 2.7, we have a definable $C^r$ map $h : X \to X_K$ as an approximation of $f$. In particular $h$ is definably homotopic to $f$. Thus by 1.1 [8], $\zeta := h^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$ and $\zeta$ is a strongly definable $C^r$ fiber bundle.

(2) Let $\zeta'$ be another strongly definable $C^r$ fiber bundle over $X$ such that $\zeta'$ is definably fiber bundle isomorphic to $\eta$. Consider the strongly definable $C^r$ fiber bundle $(\zeta, \zeta', id_X)$ whose sections represent the fiber bundle isomorphisms between $\zeta$ and $\zeta'$ which is defined in 2.11 [7]. Then it has a continuous section. By a way similar to the proof of 2.12 [7], it admits a definable $C^r$ section. This section gives a definable $C^r$ fiber bundle isomorphism between $\zeta$ and $\zeta'$. □

**Definition 2.9.** Let $G$ be a definable $C^r$ group and $0 \leq r < \infty$. Let $\Omega$ be an $n$-dimensional representation of $G$ and let $B$ be the representation map $G \to O_n(R)$ of $\Omega$. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \to B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer $k$, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:
$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, A = A', TrA = k\}$, $E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\}$, $u : E(\Omega, k) \to G(\Omega, k) : u((A, v)) = A$, where $A'$ denotes the transposed matrix of $A$ and $Tr\ A$ stands for the trace of $A$. Then $\gamma(\Omega, k)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic $G$ vector bundle. We call it *the universal $G$ vector bundle associated with $\Omega$ and $k$*. Remark that $G(\Omega, k) \subset M(\Omega)$ and $E(\Omega, k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic $G$ sets.

**Definition 2.10.** (1) Let $G$ be a definable group. A definable $G$ vector bundle $\eta = (E, p, X)$ over a definable $G$ set $X$ is called *strongly $definable$* if there exist a representation $\Omega$ of $G$ and a definable $G$ map $f : X \to G(\Omega, k)$ such that $\eta$ is definably $G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$.

(2) Let $G$ be a definable $C^r$ group and $0 \leq r \leq \infty$. A definable $C^r G$ vector bundle $\eta = (E, p, X)$ over an affine definable $C^r G$ manifold $X$ is called *strongly $definable$* if there exist a representation $\Omega$ of $G$ and a definable $C^r G$ map $f : X \to G(\Omega, k)$ such that $\eta$ is definably $C^r G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$.

To consider an equivariant version of Theorem 2.7, we need the averaging function.

Let $G = \{g_1, \ldots, g_m\}$, $X$ an affine definable $C^r G$ manifold and $\Omega$ a representation of $G$. Then we define the averaging function $A : C^r(X, \Omega) \to C^r(X, \Omega)$ by $A(f)(x) = \frac{1}{m} \sum_{i=1}^m g_i^{-1} f(g_i x)$.

Then we have the following proposition.

**Proposition 2.11.** *(1) If $f$ is a definable $C^r$ map, then $A(f)$ is a definable $C^r G$*

*map.*

*(2) Let* $Def^r(X,\Omega)$ *(resp.* $Def^r_G(X,\Omega)$*) denote the set of definable* $C^r$ *maps (resp. definable* $C^rG$ *maps) from* $X$ *to* $\Omega$*. Then* $A|Def^r_G(X,\Omega) = id_{Def^r_G(X,\Omega)}$ *and* $A(Def^r(X,\Omega)) = Def^r_G(X,\Omega)$*.*

*(3)* $A : Def^r(X,\Omega) \to Def^r(X,\Omega)$ *is continuous in the definable* $C^r$ *topology.*

As in the proof of 1.2 [9], we have the following result.

**Proposition 2.12.** *Let* $G$ *be a finite group,* $X$ *a definable* $C^rG$ *submanifold of a representation* $\Omega$ *of* $G$ *and* $1 \le r < \infty$*. Then there exists a definable* $C^rG$ *tubular neighborhood* $(U,\theta)$ *of* $X$ *in* $\Omega$*, namely* $U$ *is a* $G$ *invariant definable open neighborhood of* $X$ *in* $\Omega$ *and* $\theta : U \to X$ *is a definable* $C^rG$ *map such that* $\theta|X = id_X$*.*

By Theorem 2.7, Proposition 2.11 and 2.12, we have the following theorem.

**Theorem 2.13.** *Let* $G$ *be a finite group and* $X, Y$ *definable* $C^rG$ *submanifolds of representations* $\Omega$*,* $\Xi$ *of* $G$*, respectively and* $0 \le s < r < \infty$*. Every definable* $C^sG$ *map* $f : X \to Y$ *is approximated by a definable* $C^rG$ *map with respect to the definable* $C^s$ *topology.*

*Proof of Theorem* 1.3. Let $\eta$ be a strongly definable $G$ vector bundle over $X$. Since $\eta$ is strongly definable, there exists a definable $G$ map $f : X \to G(\Omega,\alpha)$ such that $\eta$ is definably $G$ vector bundle isomorphic to $f^*(\gamma(\Omega,\alpha))$.

By Theorem 2.13, $f$ is approximated by a definable $C^rG$ map $h : X \to G(\Omega,\alpha)$. By a way to the proof of 1.7 [10], $\eta$ is definably $G$ vector bundle isomorphic to a strongly definable $C^rG$ vector bundle $h^*(\gamma(\Omega,\alpha))$.

Let $\zeta$ be another strongly definable $C^rG$ vector bundle which is definably $G$ vector bundle isomorphic to $\eta$. Since $\eta$ and $\zeta$ are strongly definable $C^rG$ vector bundles, as in the proof of 3.1 [6], $Hom(\eta,\zeta)$ is a strongly definable $C^rG$ vector bundle. Since $\eta$ and $\zeta$ are definably $G$ vector bundle isomorphic, this definable $G$ vector bundle isomorphism defines a definable $G$ section of $Hom(\eta,\zeta)$. Using Theorem 2.13, by a way similar to [6], $\eta$ and $\zeta$ are definably $C^rG$ vector bundle isomorphic. □

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