EVERY COMPACTIFIABLE C^∞ MANIFOLD ADMITS UNCOUNTABLY MANY ALGEBRAIC MODELS

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ABSTRACT. We prove that every compactifiable C^∞ manifold M of positive dimension admits an uncountable family of nonsingular algebraic subsets $\{X_\lambda\}$ of some Euclidean space such that each X_λ is C^∞ diffeomorphic to M and that X_λ is not birationally equivalent to X_μ for $\lambda \neq \mu$.

A. Tognoli [7] proved that every closed C^∞ submanifold M of $\mathbb{R}^n$ with $2\dim M+1 \leq n$ admits a C^∞ imbedding $e: M \to \mathbb{R}^n$ arbitrarily close in the C^∞ topology to the inclusion map $M \to \mathbb{R}^n$ such that $e(M)$ is a nonsingular algebraic subset of $\mathbb{R}^n$. In particular, M has an algebraic model. J. Bochnak and W. Kucharz proved in [4] that M has a continuous family of birationally inequivalent algebraic models if M is connected and $\dim M \geq 1$. M. Shiota showed in VI.2.11 [6] that every affine Nash manifold admits an algebraic model.

In this paper, we are concerned with algebraic models of a compactifiable C^∞ manifold. Here a C^∞ manifold M is *compactifiable* if M is C^∞ diffeomorphic to the interior of a compact C^∞ manifold with boundary. We have the following result.

Theorem *Each compactifiable C^∞ manifold M of positive dimension admits an uncountable family of nonsingular algebraic subsets $\{X_\lambda\}_{\lambda\in\Lambda}$ of some Euclidean space such that every X_λ is C^∞ diffeomorphic to M and that X_λ is not birationally equivalent to X_μ for $\lambda \neq \mu$.*

The above theorem is a refinement of [5].

Proof of Theorem. By definition, we may assume that M is the interior of a compact C^∞ manifold L_1 with boundary. Moreover we may suppose that X is connected. Using [2], there exist a compact C^∞ manifold L_2 with boundary and compact submanifolds W_i $(1 \leq i \leq k)$ of Int L_2 such that ∂L_2 is C^∞ diffeomorphic to ∂L_1, each W_i intersects transverse to one another and $L_2 - W$ is C^∞ diffeomorphic to $\partial L_2 \times [0,1]$, where $W = \cup_{i=1}^k W_i$.

Let L denote the attaching space of L_1 and L_2 by the above diffeomorphism between their boundaries, regard M, L_1, L_2, W_i as C^∞ submanifolds of L. Then $L - W$ is C^∞ diffeomorphic to M. By [1], we can imbed L in some $\mathbb{R}^n$ such that L and all W_i are nonsingular algebraic subsets of $\mathbb{R}^n$. Moreover by blowing up W_i, if necessary, we may assume that each W_i is of codimension 1. Since each W_i is of codimension 1 and by Exercise P58 [3] and the proof of [4], there exist uncountably many nonsingular algebraic subsets $\{Z_\lambda\}_{\lambda\in\Lambda}$ of $\mathbb{R}^n$ fixing W such that each Z_λ is C^∞ diffeomorphic to L and that Z_λ is

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not birationally equivalent to Z_μ for $\lambda \neq \mu$. Thus each $X_\lambda := Z_\lambda - W$ is C^∞ diffeomorphic to M, it is a nonsingular algebraic set and $X_\lambda := Z_\lambda - W$ is not birationally equivalent to $X_\mu := Z_\mu - W$ for $\lambda \neq \mu$. □

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