Graph Constructions and Transfer Maps

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Abstract

This paper is expository and an outgrowth of two talks which I gave at Nagoya in Japan 2007 and at Hunan in China 2006. B.Man, E.Miller and H.Miller defined “graph construction”, which was a variant of a construction of Becker and Schultz. In this paper we generalize the graph constructions and show that they have some naturality with respect to certain transfer maps including Becker-Schultz transfer maps. Using our generalized graph construction, we re-interprets H.Miller’s stable splitting maps of Stiefel manifolds. We also describe, under some conditions, the cofiber of Becker-Schultz transfer map.

1 Introduction and statements of results

Let (X, A) be a pair of finite complexes and α be a vector bundle over X. The relative Thom complex $(X, A)^\alpha$ stands for the space $X^\alpha/(A^{\alpha|_A})$, where X^α is the usual Thom complex of α over X. We take the convention that A^α is the base point of X^α if $A = \emptyset$. Note that it has still meaning even if α is a virtual bundle: in this case X^α is a spectrum, not a space.

Let G be a compact Lie group, E a compact smooth principal G-space (E is a G-free manifold without boundary), H a closed subgroup of G and $p : E/H \to E/G$ be the bundle projection.

In this situation, Becker-Schultz [2] constructed a transfer map (a stable map)

$$t_p : (E/G)^{\zeta_G} \to (E/H)^{\zeta_H},$$

or, for a given virtual bundle α over E/G,

$$t_p : (E/G)^{\zeta_G+\alpha} \to (E/H)^{\zeta_H+p^*\alpha},$$

where, ζ_G is the vector bundle obtained by the adjoint representation of the Lie group G and the principal bundle $E \to E/G$. More precisely, let ad_G be the adjoint representation. Then $\zeta_G = \{(E \times ad_G)/G \to E/G\}$. Similarly, $\zeta_H = \{(E \times ad_H)/H \to E/H\}$.

Using ideas of Becker-Schultz [2], Man-Miller-Miller [9] constructed a map (up to homotopy) which they call “graph construction”. [1]

$$\gamma_G(E) : \mathrm{End}_G(E) \to Q((E/G)^{\zeta_G})$$

where $\mathrm{End}_G(E)$ is the set of G-equivariant self maps of E and $QX = \Omega^\infty\Sigma^\infty X$.

The following theorem is given in [9].

[1] It is natural to take the identity map of E as the base point of the set $\mathrm{End}_G(E)$, however $\gamma_G(E)$ does not seem to preserve the base points. So if necessary, adding the additional base point, we consider $\gamma_G(E) : \mathrm{End}_G(E)_+ \to Q((E/G)^{\zeta_G})$

Theorem 1.1. *The following diagram commutes (up to homotopy)*

$$\begin{array}{ccc} \mathrm{End}_G(E) & \xrightarrow{\gamma_G} & Q((E/G)^{\zeta_G}) \\ {\scriptstyle res.(G,H)}\downarrow & & {\scriptstyle \bar{t}}\downarrow \\ \mathrm{End}_H(E) & \xrightarrow{\gamma_H} & Q((E/H)^{\zeta_H}), \end{array}$$

where $\bar{t}$ is the natural map obtained by the Becker-Schultz transfer map $t_p : (E/G)^{\zeta_G} \to (E/H)^{\zeta_H}$.

In this paper we generalize the graph construction as follows:

Let (F, E) be a pair of smooth closed manifolds and G a compact Lie group which acts freely on (F, E). We denote the set of continuous maps from E to F by $\mathrm{End}(E, F)$ which has the inclusion map as the base point. Then $\mathrm{End}(E, F)$ can be seen canonically as a G-space with $\mathrm{End}_G(E, F)$ as its G-fixed point set. Let ω be the normal bundle of the inclusion $E/G \to F/G$. Let M be a compact smooth G-manifold with or without boundary. For base pointed G-spaces A and B, $Map_*^G(A, B)$ stands for the set of base point preserving G-equivariant maps from A to B.

For a space X, $\Sigma^\infty X$ denotes its associated suspension spectrum. We sometimes abbreviate $\Sigma^\infty X$ simply by X in case of no confusion.

Now we can give a parametrized graph construction which is a generalization of the graph construction:

Theorem 1.2. *Under the above notations, there exists a canonical stable map up to homotopy between spectra*

$$\gamma_G(M) : \Sigma^\infty Map_*^G(M/\partial M, \mathrm{End}(E, F)) \to (E \times_G M, E \times_G \partial M)^{p^*(\zeta_G+\omega)-\mu_M},$$

where $p : E \times_G M \to E/G$ is the bundle projection and μ_M is the bundle tangent along the fiber of p. Moreover, the map $\gamma_G(M)$ is natural for smooth G-maps: let $g : (N, \partial N) \to (M, \partial M)$ be a smooth G-map between compact G-manifolds. Then there exists a transfer map t_g such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \Sigma^\infty Map_*^G(M/\partial M, \mathrm{End}(E, F)) & \xrightarrow{\gamma_G(M)} & (E \times_G M, E \times_G \partial M)^{p^*(\zeta_G+\omega)-\mu_M} \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle t_g} \\ \Sigma^\infty Map_*^G(N/\partial N, \mathrm{End}(E, F)) & \xrightarrow[\gamma_G(N)]{} & (E \times_G N, E \times_G \partial N)^{p^*(\zeta_G+\omega)-\mu_N}. \end{array}$$

Our another result is about the cofiber of the Becker-Schultz transfer map. This result would be known to specialists, but I have never seen it in the literature.

We assume that there exists

$$\text{a } G\text{-representation } \exists U \quad \textit{such that} \quad G/H = S(U) \quad \text{as a } G\text{-space}, \tag{1.1}$$

where $S(U)$ is the unit sphere of U with a certain metric. We denote the following bundle obtained from U by λ, that is

$$\lambda \quad = \quad E \times_G U \to E/G. \tag{1.2}$$

Theorem 1.3. *Under the assumption* (1.1)*, there exists the following cofiber sequence (stable)*

$$(E/G)^{\zeta_G} \xrightarrow{t_p} (E/H)^{\zeta_H} \to (E/G)^{\zeta_G-\lambda+1} \tag{1.3}$$

Similarly, given a (virtual) bundle α over E/G, the following is also a stable cofiber sequence.

$$(E/G)^{\zeta_G+\alpha} \xrightarrow{t_p} (E/H)^{\zeta_H+p^*\alpha} \to (E/G)^{\zeta_G+\alpha-\lambda+1} \tag{1.4}$$

In this paper we use various properties of transfer maps [3][2]. In [9] there is an excellent summary about transfer maps.

The author thanks M. Imaoka for his kind explanation to me about his note [7].

2 Relative graph construction

We denote by $\widetilde{\Sigma}Y$ the unreduced suspension of Y. The base point of $\widetilde{\Sigma}Y$ is assumed to be $([0,*])$. Let H be a closed subgroup of a compact Lie group G. Consider the G-equivariant (based) cofiber sequence

$$(G/H)_+ \to (G/G)_+ \to \widetilde{\Sigma}(G/H) \to \Sigma(G/H)_+$$

For a based G-space X, applying $Map_*^G(\ ,X)$ to the above cofiber sequence, we have the fiber sequence

$$\Omega(X^H, X^G) \to X^G \to X^H,$$

where X^G is the G-fixed point set of X.

Now consider the special cases of Theorem 1.3.

When $M = G/G = \{\,1pt.\}$, then $\mu_M = 0$, $p = id$ and $Map_*^G(\{\,1pt.\}_+, \mathrm{End}(E,F)) = \mathrm{End}_G(E,F)$. So in this case we have a stable map between of spectra.

$$\gamma_G(E,F) : \mathrm{End}_G(E,F) \to (E/G)^{\zeta_G+\omega}, \tag{2.1}$$

which we call the *relative graph construction*. Note that the relative graph construction can be written as a homotopy class between spaces:

$$\gamma_G(E,F) : \mathrm{End}_G(E,F) \to Q((E/G)^{\zeta_G+\omega}). \tag{2.2}$$

By construction, it is easy to see that the relative construction $\gamma_G(E,E)$ in case of $E = F$ is equal to the original graph construction $\gamma_G(E)$.

If we take G/H as M, then $Map_*^G(G/H_+, \mathrm{End}(E,F)) = \mathrm{End}_H(E,F)$. Applying Theorem 1.2, we obtain the stable map $\gamma_H : \mathrm{End}_H(E,F) \to (E/H)^{\zeta_H+\omega_H}$. The naturality of the Theorem 1.2 gives Theorem 1.1 for the case $E = F$.

By Theorem 1.2 and Theorem 1.3, we obtain the following.

Corollary 2.1. *Suppose that* (G,H) *satisfies the condition* (1.1). *Then there exists a stable map*

$$\widetilde{\gamma} : \frac{\mathrm{End}_H(E,F)}{\mathrm{End}_G(E,F)} \to (E/G)^{\zeta_G+\omega_G-\lambda+1},$$

which satisfies the obvious commutativity:

$$\begin{array}{ccccccc}
\mathrm{End}_G(E,F) & \longrightarrow & \mathrm{End}_H(E,F)) & \longrightarrow & \dfrac{\mathrm{End}_H(E,F)}{\mathrm{End}_G(E,F)} & \longrightarrow & \Sigma\,\mathrm{End}_G(E,F) \\
{\scriptstyle \gamma_G(E,F)}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_H(E,F)} & & \big\downarrow{\scriptstyle \tilde{\gamma}} & & {\scriptstyle \Sigma\gamma_G(E,F)}\big\downarrow \\
(E/G)^{\zeta_G+\omega_G} & \xrightarrow{t} & (E/H)^{\zeta_H+\omega_H} & \longrightarrow & (E/G)^{\zeta_G+\omega_G-\lambda+1} & \longrightarrow & \Sigma(E/G)^{\zeta_G+\omega_G},
\end{array}$$

where the both holizontal lines are stable cofiber sequences.

Relative graph constructions have various (obvious) naturalities. We summarize:

Proposition 2.2. *The following three diagrams* $(1)-(3)$ *are all commutative up to homotopy.*

$$(1)\qquad \begin{array}{ccc} \mathrm{End}_G(E,F) & \xrightarrow{res.(G,H)} & \mathrm{End}_H(E,F) \\ \gamma_G(E,F)\big\downarrow & & \big\downarrow\gamma_H(E,F) \\ Q(E/G)^{\zeta_G+\omega_G} & \xrightarrow{t'} & Q(E/H)^{\zeta_H+\omega_H}, \end{array}$$

where, t' is the transfer map of $p: E/H \to E/G$ and the bundle $\zeta_G+\omega_G$ over E/G, ω_G is the normal bundle of the inclusion $E/G \to F/G$, and ω_H is the normal bundle of the inclusion $E/H \to F/H$.

$$(2)\qquad \begin{array}{ccc} \mathrm{End}_G(E,F) & \xrightarrow{(inc.)_*} & \mathrm{End}_G(E,F') \\ \gamma_G(E,F)\big\downarrow & & \big\downarrow\gamma_G(E,F') \\ Q(E/G)^{\zeta_G+\omega_G} & \xrightarrow{i'} & Q(E/G)^{\zeta_G+\omega'_G}, \end{array}$$

where $E \subseteq F \subseteq F'$ are smooth manifolds on which G acts freely, ω_G is the normal bundle of the inclusion $E/G \to F/G$, ω'_G is the normal bundle of the inclusion $E/G \to F'/G$ and i' is the inclusion map.

$$(3)\qquad \begin{array}{ccc} \mathrm{End}_G(E,F) & \xrightarrow{(inc.)^*} & \mathrm{End}_G(E',F) \\ \gamma_G(E,F)\big\downarrow & & \big\downarrow\gamma_G(E',F) \\ Q(E/G)^{\zeta_G+\omega_G(E)} & \xrightarrow{t'} & Q(E'/G)^{\zeta_G+\omega_G(E')}, \end{array}$$

where $E' \subseteq E \subseteq F$ are smooth manifolds on which G acts freely, t' is the transfer map of the inclusion map $E'/G \to E/G$ and the bundle $\zeta_G+\omega_G(E)$ over E/G: $\omega_G(E)$ and $\omega_G(E')$ are the normal bundles of the inclusions E/G and E'/G to F/G, respectively.

There is an useful property of the graph construction in page 243 of [9]. The following proposition is a variant of it.

Proposition 2.3. *Let M be a compact manifold with or without boundary ∂M and with trivial G action. Let $i: E \to F$ be the inclusion and $\Delta': E \to E \times F$ defined by $\Delta'(x) = (x, i(x))$. Suppose that there exists a map $f_1 : M/\partial M \to \mathrm{End}_G(E,F)$ such that*

1. *the resulting equivariant map $f: M \times E \to F$ is smooth, where $f(m,e) = (f_1([m])(e)$.*
2. *reduced graph $f'/G : M \times B = M \times_G E \to E \times_G F$ given by $f'(m,e) = (e, f(m,e))$ is transverse to $\Delta'/G : E/G \to E \times_G F$, moreover we assume that $(\partial M \times_G F) \cap f'^{-1}(Im\,\Delta'/G) = \emptyset$.*

Consider the following pull-back diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{g} & E/G \\ \big\downarrow & & \big\downarrow\Delta'/G \\ M \times (E/G) & \xrightarrow[f'/G]{} & E \times_G F, \end{array}$$

Denote the composite $\Gamma \to M \times E/G \xrightarrow{proj.} M$ by p. Then the following diagram commutes up to homotopy:

$$\begin{array}{ccc} M/\partial M & \xrightarrow{t_p} & Q(\Gamma^{g^*\zeta_G+\omega_G}) \\ \big\downarrow f_1 & & \big\downarrow Q(\bar{g}) \\ \mathrm{End}_G(E) & \xrightarrow{\gamma_G} & Q((E/G)^{\zeta_G+\omega_G}), \end{array}$$

where t_p is the transfer map of p and $Q(\bar{g})$ is the canonical map induced by g.

3 Miller's splitting map

In this section we will give an interpretation of the splitting map of Miller's stable decomposition [10], using the relative graph construction.

We denote the real numbers by $\mathbb{R}$, the complex numbers by $\mathbb{C}$ and the quaternions by $\mathbb{H}$.

According as $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, let $G_{\mathbb{F}}(n) = O(n), U(n), Sp(n)$ respectively.

Fixing the field $\mathbb{F}$, let G_n be $G_{\mathbb{F}}(n)$ and $V_{q,k} = G_q/G_{q-k}$ be the Stiefel manifold over $\mathbb{F}$. On $V_{q,k}$, G_q acts from the left and G_k acts from the right. The both action are consistent. For the Stiefel manifold $F = V_{n,k}$, F/G_k is the Grassmann manifold $G_{n,k}$. Let ζ_k be the adjoint bundle over $G_{n,k}$ associated with G_k and ξ_k be the canonical k-dimensional bundle over $G_{n,k}$.

H.Miller's stable decomposition of Stiefel manifolds [10] (See also [4] [1].)

$$V_{n,q}^{+} = \bigvee_{k=0}^{q} G_{q,k}^{\zeta_k+(n-q)\xi_k}$$

can be explained as follows.
Let $0 \le k \le q$,

1. The normal bundle of the inclusion $G_{q,k} \to G_{n,k}$ is isomorphic to $(n-q)\xi_k$, because of the existence of an open imbedding $V'_{q,k} \times \mathrm{Hom}\,(\mathbb{F}^k, \mathbb{F}^{n-q}) \to V'_{n,k}$, where $V'_{n,k}$ is the Stiefel manifold , consisting of k independent vectors of $\mathbb{F}^n$. It is easy to see that $(n-q)\xi_k = V_{m,k} \times_{G_k} \mathrm{Hom}\,(\mathbb{F}^k, \mathbb{F}^{n-q})$ is homeomorphic to $V'_{m,k} \times_{G'_k} \mathrm{Hom}\,(\mathbb{F}^k, \mathbb{F}^{n-q})$, where $G'_k = GL(k, \mathbb{F})$

2. Consider the relative graph construction,
   $\gamma = \gamma_{G_k}(V_{q,k}, V_{n,k}) : \mathrm{End}_{G_k}(V_{q,k}, V_{n,k})_+ \to Q((V_{q,k}/G_k)^{\zeta_{G_k}+\omega})$, so we have ,
   $$\gamma : \mathrm{End}_{G_k}(V_{q,k}, V_{n,k})_+ \to Q(G_{q,k}^{\zeta_k+(n-q)\xi_k})$$

3. There exists a natural map
   $$f_1 : V_{n,q} \to \mathrm{End}_{G_k}(V_{q,k}, V_{n,k}),$$
   this map corresponds to the left multiplication of the matrices.

Therefore we have,

$$s_k : V_{n,q}^{+} \to \mathrm{End}_{G_k}(V_{q,k}, V_{n,k})_+ \to Q(G_{q,k}^{\zeta_k+(n-q)\xi_k}),$$

which is the desired retraction map.

To see this, consider the map $f = adj(f_1) : V_{n,q} \times V_{q,k} \to V_{n,k}$ and $f' : V_{n,q} \times V_{q,k} \to V_{q,k} \times V_{n,k}$ by $f'(x, y) = (y, f(x, y))$. Define the space Γ by the following pull-back diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ l\ } & G_{q,k} = V_{q,k}/G_k \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle (\Delta' = id \times i_0)/G_k} \\ V_{n,q} \times G_{q,k} & \xrightarrow{f'/G_k} & (V_{q,k} \times V_{n,k})/G_k, \end{array}$$

Γ is just the $\Gamma_{n,q,q-k}$ in H.Miller's notation(his φ_0 is our $-i_0$). Now according to Man-Miller-Miller's p243 and H.Miller's Proposition 3.3, as we cite in Proposition 2.3, we see that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} V_{n,q}^{+} & \xrightarrow{\ t_p\ } & Q(\Gamma^{l^*(\zeta_k+(n-q)\xi_k)}) \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle Q(\bar{l})} \\ \mathrm{End}_{G_k}(V_{q,k}, V_{n,q})_+ & \xrightarrow{\ \gamma\ } & Q(G_{q,k}^{\zeta_k+(n-q)\xi_k}), \end{array}$$

where p is the composite $\Gamma \xrightarrow{j} V_{n,q} \times G_{q,k} \xrightarrow{p_1} V_{n,q}$ and t_p is the transfer with respect to p. By construction, we see the composite $V_{n,q}^{+} \xrightarrow{t_p} Q(\Gamma^{l^*(\zeta_k+(n-q)\xi_k)}) \to Q(G_{q,k}^{\zeta_k+(n-q)\xi_k})$ is just the splitting map $s_k : V_{n,q}^{+} \to Q(G_{q,k}^{\zeta_k+(n-q)\xi_k})$ in H. Miller's notation [10].

So we see that our map constructed by relative graph construction is the precisely the Miller's splitting map s_k.

4 $R_* : \pi_*(U(n)) \to \pi_*(O(2n))$

Let $R : U(n) \to O(2n)$ be the realification map. To study the induced homomorphism R_* between the homotopy groups in the meta-stable range, it is important to know the following composite homomorphism (from the upper left to the lower right):

$$\begin{array}{ccccccc}
\pi_*^s(\Sigma\mathbb{C}P_n^\infty) & & & & & & \\
\cong\Big\uparrow E^\infty & & & & & & (\sharp) \\
\pi_*(\Sigma\mathbb{C}P_n^\infty) & \underset{r_*}{\xrightarrow{\cong}} & \pi_*(U(\infty)/U(n)) & \xrightarrow{R_*} & \pi_*(O(\infty)/O(2n)) & \underset{r_*}{\xleftarrow{\cong}} & \pi_*(\mathbb{R}P_{2n}^\infty) \\
 & & & & & & \cong\Big\downarrow E^\infty \\
 & & & & & & \pi_*^s(\mathbb{R}P_{2n}^\infty),
\end{array}$$

where r's are reflection maps and $\mathbb{C}P_n^\infty = \mathbb{C}P^\infty/\mathbb{C}P^{n-1}$ is the stunted complex projective space and $\mathbb{R}P_{2n}^\infty$ is the real stunted projective spaces. Note that in the "meta-stable range", the both r_*'s and E^∞'s are isomorphic. we show that

Proposition 4.1. *There exists a stable map $t : \Sigma\mathbb{C}P_n^\infty \to \mathbb{R}P_{2n}^\infty$ whose cofiber is the stable Thom complex $(\mathbb{C}P^\infty)^{n\xi+2-\xi^2}$ and t induces the composite homomorphisms of $(\sharp)$, where ξ is the complex canonical line bundle, ξ^2 means the tensor product over $\mathbb{C}$.*

Proof. James [6] showed that there exists a map $\theta : G_{\mathbb{F}}(n) \to Q(Q_{\mathbb{F}}^n)$ such that $\theta \circ r = \pm E^\infty$, where $Q_{\mathbb{F}}^n$ is the $\mathbb{F}$ quasi-projective space and $r : Q_{\mathbb{F}}^n \to G_{\mathbb{F}}(n)$ is the reflection map.

Let $G = G_{\mathbb{F}}(1)$. Then $Q_{\mathbb{F}}^n$ is equal to the Thom complex $(S(\mathbb{F}^n)/G)^{\zeta_G}$, where $S(\mathbb{F}^n)$ is the unit sphere in $\mathbb{F}^n$.

Some people including Becker-Schultz[2], M. Crabb[5] or Man-Miller-Miller[9] showed that the James splitting map θ can be taken as the composite

$$G_{\mathbb{F}}(n) \to \mathrm{End}_G(S(\mathbb{F}^n)) \xrightarrow{\gamma} Q(S(\mathbb{F}^n)/G)^{\zeta_G}) = Q(Q_{\mathbb{F}}^n),$$

here γ is the graph construction. Recall that

$$Q_{\mathbb{F}}^n = \begin{cases} \Sigma\mathbb{C}P_+^{n-1} & \text{for } \mathbb{F} = \mathbb{C} \\ \mathbb{R}P_+^{n-1} & \text{for } \mathbb{F} = \mathbb{R}. \end{cases}$$

Since the graph construction has the naturality as in Theorem 1.1 with Becker-Schultz transfer maps, we have the following commutative diagram.

$$\begin{array}{ccc}
\Sigma\mathbb{C}P_+^{n-1} & & \mathbb{R}P_+^{2n-1} \\
\Big\downarrow r_{\mathbb{C}} & & \Big\downarrow r_{\mathbb{R}} \\
U(n) & \xrightarrow{R} & O(2n) \\
\Big\downarrow \theta_{\mathbb{C}} & & \Big\downarrow \theta_{\mathbb{R}} \\
\Omega^\infty\Sigma^\infty\Sigma\mathbb{C}P_+^{n-1} & \xrightarrow{t} & \Omega^\infty\Sigma^\infty\mathbb{R}P_+^{2n-1},
\end{array}$$

where $\theta_{\mathbb{F}} \circ r_{\mathbb{F}} = \pm E^{\infty}$ and t is the Becker-Schultz transfer map. Since all maps in the above diagram are compatible with respect to n, we have the commutative diagram

$$\begin{array}{ccc} \Sigma\mathbb{C}P_n^{\infty} & & \mathbb{R}P_{2n}^{\infty} \\ \downarrow{\scriptstyle r_{\mathbb{C}}} & & \downarrow{\scriptstyle r_{\mathbb{R}}} \\ U(\infty)/U(n) & \xrightarrow{R} & O(\infty)/O(2n). \\ \downarrow{\scriptstyle \theta_{\mathbb{C}}} & & \downarrow{\scriptstyle \theta_{\mathbb{R}}} \\ \Omega^{\infty}\Sigma^{\infty}\Sigma\mathbb{C}P_n^{\infty} & \xrightarrow{t} & \Omega^{\infty}\Sigma^{\infty}\mathbb{R}P_{2n}^{\infty} \end{array}$$

In the meta-stable range, r_* induces the isomorphism between the homotopy groups and also the suspension E^{∞} induces the isomorphism. Remark that the above θ's in the last diagram can be considered as the Miller's splitting map s_1.

Now the rest of the proof easily follows from Theorem 1.3 and the following observations.

Let $E = S(\mathbb{C}^n)$ and suppose that $G = S^1$ acts on E by scalar multiplication. Let $H = Z/2$, then $U = \mathbb{C}$, where the action of G on U is given by $x \cdot z = x(z^2)$ for $x \in \mathbb{C}$ and $z \in S^1$. In this case $\lambda = \xi^2$, where ξ is the canonical line bundle over $\mathbb{C}P$, we get the stable cofiber sequence

$$\Sigma\mathbb{C}P_+^{n-1} \xrightarrow{t} \mathbb{R}P_+^{2n-1} \to (\mathbb{C}P^{n-1})^{2-\xi^2}. \tag{4.1}$$

This completes the proof. □

Remark 4.2. In the case $(\mathbb{H}, \mathbb{C})$, let $E = S(\mathbb{H}^n)$ and let $G = S^3$ act on E by the scalar multiplication. Let $H = S^1$, then, since $S^3/S^1 = S(ad_{S^3})$, in this case $U = ad_{S^3}$ and $\lambda = \zeta_G$, we get the cofiber sequence

$$Q^n \xrightarrow{t} \Sigma\mathbb{C}P_+^{2n-1} \to \Sigma\mathbb{H}P_+^{n-1}$$

Note that that this cofiber sequence exists unstably (without suspension). On the other hand, in the case $(\mathbb{C}, \mathbb{R})$ the sequence (4.1) would not exists unstably.

5 The proof of Theorem 1.3 and 1.4

First we give the construction of the stable map $\gamma_G(M)$.

1. Take an imbedding $i : (E \times M)/G \to \mathbb{R}^k$ (resp. D^k). We denote its normal bundle by $\nu = \nu_M$. Using Pontrjagin construction, we have a map $c : S^k \to (E \times_G M, E \times_G \partial M)^{\nu}$.

2. For a map $f : E \times M \to F$ (G-equivariant map which is NOT necessary to be smooth.), take its graph $f' : E \times M \to E \times M \times F$, defined by $f'(x, y) = (x, y, f(x, y))$. Dividing by G, we have the map
$$f'/G : ((E \times_G M), (E \times_G \partial M))^{\nu} \to ((E \times M \times F)/G), (E \times \partial M \times F)/G)^{q*\nu}$$
between the Thom complexes, where the map $q : (E \times M \times F)/G \to (E \times M)/G$ is induced by the projection map to the first 2 factors.

3. (This construction does not depend on the map f.) Consider the map $\Delta' : E \times M \to E \times M \times F$ defined by $\Delta'(x, y) = (x, y, i(x))$. Then the normal bundle of $\Delta'/G : (E \times M)/G \to (E \times M \times F)/G$ is isomorphic to $p^*(\tau(E)/G + \omega)$, where $p : (E \times M)/G \to E/G$ is the bundle projection.

We denote the bundle tangent along the fiber of p by $\mu = \mu_M$. Then, $\tau(E)/G = \tau(E/G) + \zeta_G$ and $p^*(\tau(E/G)) = \tau(E \times_G M) - \mu$, where $\tau(X)$ is the tangent bundle of a manifold X.

Consider the Pontrjagin construction about the imbedding

$$E \times_G M \xrightarrow{\Delta'/G} (E \times M \times F)/G \xrightarrow{zero-section} q^*\nu,$$

we have the (relative) umkehr map

$$\begin{aligned} t_{\Delta'} &: ((E \times M \times F)/G, (E \times \partial M \times F)/G)^{q*\nu} \\ &\to (E \times_G M, E \times_G \partial M)^{\nu + (\tau(E \times_G M) - \mu + p^*(\zeta_G + \omega))} = \Sigma^k (E \times_G M, E \times_G \partial M)^{p^*(\zeta_G+\omega)-\mu} \end{aligned}$$

4. Composing previous maps, we get the map

$$\begin{aligned} S^k \xrightarrow{c} (E \times_G M, E \times_G \partial M)^{\nu} &\to ((E \times M \times F)/G, (E \times \partial M \times F)/G)^{q*\nu} \\ &\to \Sigma^k (E \times_G M, E \times_G \partial M)^{p^*(\zeta_G+\omega)-\mu}, \end{aligned}$$

where c is the Pontrjagin construction.

Thus, we obtain a stable map

$$\gamma_G(M) : \Sigma^\infty Map_*^G(M/\partial M, \mathrm{End}(E,F)) \to (E \times_G M, E \times_G \partial M)^{p^*(\zeta_G+\omega)-\mu_M}$$

Note that $M/\partial M = M^+$ in the case that $\partial M = \emptyset$.

Next we give the proof of naturality.

The proof is almost the same as in the proof of Theorem 3.4 in [9]. For simplicity, we will prove only in the case $\partial M = \emptyset$ and $\partial N = \emptyset$. We have a homotopy-commutative diagram

$$\begin{array}{ccccccc} S^k & \xrightarrow{c} & (E \times_G M)^{\nu_M} & \xrightarrow{f'/G} & ((E \times M \times F)/G)^{q*\nu_M} & \xrightarrow{t_{\Delta'}} & \Sigma^k (E \times_G M)^{p^*(\zeta_G+\omega)-\mu_M} \\ \| & & \downarrow t_g & & \downarrow t_g & & \downarrow t_g \\ S^k & \xrightarrow{c} & (E \times_G N)^{\nu_N} & \xrightarrow{(f \circ g)'/G} & ((E \times N \times F)/G)^{q*\nu_N} & \xrightarrow{t_{\Delta'}} & \Sigma^k (E \times_G N)^{p^*(\zeta_G+\omega)-\mu_N} \end{array}$$

from which the theorem follows.

Proof of Theorem 1.4.

By (1.1) and (1.2) we have

$$p^*\lambda = 1 + \tau_p, \tag{5.1}$$

where τ_p is the bundle tangent along the fiber of $p : E/H \to E/G$.

Under the assumption (1.1), $E/H = E \times_G (G/H)$ is the sphere bundle of λ, i.e.,

$$E/H = S(\lambda).$$

Let α and β be vector bundles over B. Then the following sequence is a cofiber sequence: (See James's book [6] page 36)

$$S(\alpha)^{p^*\beta} \to B^\beta \xrightarrow{j} B^{\alpha+\beta} \xrightarrow{\partial} \Sigma S(\alpha)^{p^*\beta} = S(\alpha)^{1+p^*\beta} \tag{5.2}$$

Even if the above β is a virtual bundle, (5.2) has a meaning in the stable homotopy category and it is still the cofiber sequence.

Consider the case that $B = E/G$, $\alpha = \lambda$ and $\beta = \zeta_G - \lambda$,

$$B^{\alpha+\beta} = B^{\zeta_G} = (E/G)^{\zeta_G},$$

$$S(\alpha)^{1+p^*\beta} = S(\lambda)^{1+p^*(\zeta_G-\lambda)} = S(\lambda)^{\zeta_H+\tau_p+1-p^*\lambda} = S(\lambda)^{\zeta_H} = (E/H)^{\zeta_H},$$

Thus the above ∂ gives a stable map of Becker-Schultz type. It is a folklore theorem: Let λ (resp.β) be a (resp. virtual bundle) bundle over B. The umkehr map (See [3] and [9]) $t : B^{\lambda\oplus\beta} \to \Sigma S(\lambda)^{p^*\beta}$ of the sphere bundle $S(\lambda) \xrightarrow{p} B$ is just equal to the connecting map ∂ of the Gysin sequence (5.2) up to sign [7] [8]. In our case, by construction, this umkehr map coincides with the Becker-Schultz transfer.

References

[1] G. Arone, The Mitchell-Richter filtration of loops on Stiefel manifolds stably splits, Proc. Amer. Math. Soc. 129(2001), 1207–1211.

[2] J. C. Becker and R. E. Schultz, Equivariant function spaces and stable homotopy theory I, Comm. Math. Helv. 49(1974), 1–34

[3] M. J. Boardman, Stable homotopy theory, Chapter V - Duality and Thom spectra, mimeographed. University of Warwick, 1966.

[4] M. C. Crabb, On the stable splitting of $U(n)$ and $\Omega U(n)$, Lecture Note in Math., Springer-Verlag, 1298(1986), 35–53.

[5] M. C. Crabb, $\mathbb{Z}/2$-Homotopy Theory, London Math. Soc. Lecture Notes ser. 44(1980).

[6] I. James, Topology of Stiefel manifolds, Lecture Notes Series ,London Math. Soc. 76(1976)

[7] M. Imaoka, Studies by transfer maps on geometric properties of bundles and its application, Report of JSPS (Japanese), 2002.

[8] K. Knapp, Some applications of K-theory to framed bordism: e-invariant and transfer, Habilitationsschrift, Bonn, 1979.

[9] B. M. Mann, E. Y. Miller and H. R. Miller, S^1-equivariant function spaces and characteristic classes, Trans. A.M.S. 295(1986) 233–256

[10] H. R. Miller, Stable splittings of Stable manifolds, Topology, 24(1985), 150–171