Relative properties of definable C^rG manifolds

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Abstract

Let G be a compact definable C^r group and $1 \leq r < \infty$.

We prove that if X is a noncompact affine definable C^rG manifold and $X_1, \dots, X_n$ are noncompact definable C^rG submanifolds of X in general position such that $(X; X_1, \dots, X_n)$ satisfies the frontier condition, then $(X; X_1, \dots, X_n)$ is simultaneously definably C^rG compactifiable.

Moreover we prove that if $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$) are definable C^rG submanifolds of an affine definable C^rG manifold X (resp. Y) in general position, then every definable C^1G map $(X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ is approximated by a definable C^rG map $(X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$.

Furthermore we prove that we can raise simultaneously differentiability of a definable C^2 manifold and its definable C^2 submanifolds such that they satisfy some condition.

2000 *Mathematics Subject Classification.* 14P10, 14P20, 58A05, 03C64.
Keywords and Phrases. O-minimal, definable C^rG manifolds, simultaneous definable C^rG compactifications, simultaneous definable C^rG approximations, relative structures.

1 . Introduction.

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. General references on o-minimal structures are [2], [5], see also [15]. It is known in [14] that there exist uncountably many o-minimal expansions of $\mathcal{R}$. For example, the Nash category is a special case of the definable C^r category and it coincides with the definable C^∞ category based on $\mathcal{R} = (\mathbb{R}, +, \cdot, >)$ [16]. Further properties and constructions of them are studied in [3], [4], [6], [13]. Equivariant definable category is studied in [8], [9], [10], [12].

In this paper "definable" means "definable with parameters in $\mathcal{M}$", everything is considered in $\mathcal{M}$, every manifold does not have boundary, each definable map is continuous and $1 \leq r < \infty$ unless otherwise stated.

Let X be a C^r manifold and $X_1, \dots, X_n$ C^r submanifolds of X. We say that $X_1, \dots, X_n$ *are in general position* in X if for each $i \in \{1, \dots, n\}$ and $J \subset \{1, \dots, n\} - \{i\}$, X_i intersects transverse to $\cap_{j \in J} X_j$.

Let G be a compact definable C^r group, X a noncompact definable C^rG manifold and $X_1, \dots, X_n$ noncompact definable C^rG submanifolds of X in general position. If X is affine, then by 2.10 [10], we may assume that X is a bounded definable C^rG submanifold of some representation Ω of G. We say that $(X; X_1, \dots, X_n)$ satisfies *the frontier condition* if each $\overline{X_i} - X_i$ is contained in $\overline{X} - X$, where $\overline{X_i}$ (resp. $\overline{X}$) denotes the closure of X_i (resp. X) in Ω. We say that $(X; X_1, \dots, X_n)$ is *simultaneously definably C^rG compactifiable* if there exist a compact definable C^rG manifold Y with boundary ∂Y, compact definable C^rG submanifolds $Y_1, \dots, Y_n$ of Y with boundary $\partial Y_1, \dots, \partial Y_n$, respectively, and a definable C^rG diffeomorphism $f : X \to Int\ Y$ such that for any i, $f(X_i) = Int\ Y_i$, each ∂Y_i is contained in ∂Y, and $Y_1, \dots, Y_n$ and ∂Y are in general position in Y. Here $Int\ Y$ (resp. $Int\ Y_i$) denotes the interior of Y (resp. Y_i).

Theorem 1.1. *Let G be a compact definable C^r group, X a noncompact affine definable C^rG manifold and $X_1, \dots, X_n$ noncompact definable C^rG submanifolds of X in general position such that $(X; X_1, \dots, X_n)$ satisfies the frontier condition. Then $(X; X_1, \dots, X_n)$ is simultaneously definably C^rG compactifiable.*

Theorem 1.1 is a relative version of 1.2 [10].

Let G be a compact definable C^r group. Let X be a definable C^rG manifold with boundary ∂X and $X_1, \dots, X_n$ definable C^rG submanifolds of X with boundary $\partial X_1, \dots, \partial X_n$, respectively, such that every ∂X_i is contained in ∂X. A *relative definable C^rG collar* of $(\partial X; \partial X_1, \dots, \partial X_n)$ is a definable C^rG imbedding $\phi : (\partial X \times [0,1]; \partial X_1 \times [0,1], \dots, \partial X_n \times [0,1]) \to (X; X_1, \dots, X_n)$ such that $\phi|\partial X \times \{0\}$ is the inclusion $\partial X \to X$, where the action on $[0,1]$ is trivial.

Theorem 1.2. *Let G be a compact definable C^r group. Let X be a compact affine definable C^rG manifold with boundary ∂X, and $X_1, \dots, X_n$ compact definable C^rG submanifolds of X with boundary $\partial X_1, \dots, \partial X_n$, respectively, such that $X_1, \dots, X_n, \partial X$ are in general position, every ∂X_i is contained in ∂X and $2 \le r < \infty$. Then there exists a relative definable C^rG collar $\phi : (\partial X \times [0,1]; \partial X_1 \times [0,1], \dots, \partial X_n \times [0,1]) \to (X; X_1, \dots, X_n)$ of $(\partial X; \partial X_1, \dots, \partial X_n)$.*

Theorem 1.2 is a relative version of 4.6 [9].

Let $Def^r(\mathbb{R}^n)$ denote the set of definable C^r functions on $\mathbb{R}^n$. For each $f \in Def^r(\mathbb{R}^n)$ and for each positive definable function $\epsilon : \mathbb{R}^n \to \mathbb{R}$, the ϵ-neighborhood $N(f;\epsilon)$ of f in $Def^r(\mathbb{R}^n)$ is defined by $\{h \in Def^r(\mathbb{R}^n) | |\partial^\alpha(h - f)| < \epsilon, \forall \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \le r\}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n, |\alpha| = \alpha_1 + \dots + \alpha_n, \partial^\alpha F = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. We call the topology defined by these ϵ-neighborhoods the *definable C^r topology*. By taking relative topology, we can define the *definable C^r topology* on a definable C^r submanifold of $\mathbb{R}^n$.

The following is a relative version of 1.1 [9].

Theorem 1.3. *Let G be a compact definable C^r group. Let X, Y be affine definable C^rG manifolds and $X_1, \dots, X_n$ (reps. $Y_1, \dots, Y_n$) definable C^rG submanifolds of X (resp Y) such that $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$) are in general position. Suppose that $f : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ is a definable C^1G map. Then f is approximated by a definable C^rG map $h : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ in the definable C^1 topology. Moreover if for $1 \le i_1 < \dots < i_k \le n$, $f|X_{i_1}, \dots, f|X_{i_k}$ are definable C^rG maps, then we can take h such that $h| \cup_{j=1}^k X_{i_j} = f| \cup_{j=1}^k X_{i_j}$.*

The following proposition is obtained by II.5.3 [15] and II.5.11 [15].

Theorem 1.4 ([15]). *Let X, Y be definable C^r submanifolds of $\mathbb{R}^n$. Let $f : X \to Y$ be a definable C^r map. If f is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image), then an approximation of f in the definable C^r topology is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image). Moreover if f is a diffeomorphism, then $h^{-1} \to f^{-1}$ as $h \to f$.*

Let X be a definable C^s manifold, $X_1, \ldots, X_n$ definable C^s submanifolds of X and $1 \leq s < r \leq \omega$. A *simultaneous definable C^r manifold structure* $(Y; Y_1, \ldots, Y_n)$ of $(X; X_1, \ldots, X_n)$ is a definable C^r manifold Y and definable C^r submanifolds $Y_1, \ldots, Y_n$ of Y such that there exists a definable C^s diffeomorphism $(X; X_1, \ldots, X_n) \to (Y; Y_1, \ldots, Y_n)$.

Theorem 1.5. *Let X be a definable C^2 manifold, $X_1, \ldots, X_n$ definable C^2 submanifolds of X in general position and $2 \leq r < \infty$. If either $X, X_1, \ldots, X_n$ are compact or $X, X_1, \ldots, X_n$ are noncompact and $(X; X_1, \ldots, X_n)$ satisfies the frontier condition, then $(X; X_1, \ldots, X_n)$ admits a simultaneous definable C^r manifold structure $(\tilde{X}; \tilde{X}_1, \ldots, \tilde{X}_n)$. Moreover if $(\hat{X}; \hat{X}_1, \ldots, \hat{X}_n)$ is another simultaneous definable C^r manifold structure of $(X; X_1, \ldots, X_n)$, then $(\tilde{X}; \tilde{X}_1, \ldots, \tilde{X}_n)$ is definably C^r diffeomorphic to $(\hat{X}; \hat{X}_1, \ldots, \hat{X}_n)$.*

Theorem 1.5 is a relative version of 1.2 [11].

2 Proof of our results

Let G be a compact definable C^r group. A *representation map* of G is a group homomorphism from G to some orthogonal group which is a definable C^r map. A *representation* means the representation space of a representation map of G. In this paper, we assume that every representation of G is orthogonal.

Definable C^rG manifolds are studied in [9], [10], [12]. A *definable C^rG submanifold* of a representation Ω of G is a G invariant definable C^r submanifold of Ω. A definable C^rG manifold is *affine* if it is definably C^rG diffeomorphic to a definable C^rG submanifold of some representation of G.

Let f be a G invariant surjective submersive definable C^r map from a definable C^rG manifold S to a definable C^r manifold A. We say that f is *piecewise definably C^rG trivial* if there exists a finite partition of A into definable C^r submanifolds C_i of A such that for each C_i there exists a definable C^rG diffeomorphism $k_i : f^{-1}(C_i) \to C_i \times f^{-1}(a_i)$ with $f|f^{-1}(C_i) = p_i \circ k_i$, where $a_i \in C_i$ and p_i denotes the projection $C_i \times f^{-1}(a_i) \to C_i$.

Theorem 2.1 (1.1 [10])**.** *Let G be a compact definable C^r group. Let S be a definable C^rG submanifold of a representation of G and let A be a definable C^r submanifold of $\mathbb{R}^n$. Then every G invariant surjective submersive definable C^r map $f : S \to A$ is piecewise definably C^rG trivial.*

By a way similar to the proof of 2.10 [10], we have the following proposition.

Proposition 2.2. *Let G be a compact definable C^r group, X a noncompact affine definable C^rG manifold and $X_1, \ldots, X_n$ noncompact definable C^rG submanifolds of X in general position such that $(X; X_1, \ldots, X_n)$ satisfies the frontier condition. Then we may assume that X is a bounded definable C^rG submanifold of some representation Ω of G such that $\overline{X_1} - X_1 = \cdots = \overline{X_n} - X_n = \overline{X} - X = \{0\}$, where $\overline{X}$ (resp. $\overline{X_i}$) denotes the closure of X (resp. X_i) in Ω.*

Theorem 2.3 (4.6 [9])**.** *Let G be a compact definable C^r group, X a compact affine definable C^rG manifold with boundary ∂X and $2 \leq r < \infty$. Then there exists a definable C^rG collar, namely there exists a definable C^rG imbedding $\phi : \partial X \times [0,1] \to X$ such that $\phi|(\partial X \times \{0\})$ is the inclusion $\partial X \to X$, where the action on $[0,1]$ is trivial.*

Theorem 2.4 (1.2 [9])**.** *If G is a compact definable C^r group, then every definable C^rG submanifold X of a representation Ω of G has a definable C^rG tubular neighborhood (U, θ) of X in Ω, namely U is a G invariant definable open neighborhood of X in Ω and $\theta : U \to X$ is a definable C^rG map with $\theta|X = id_X$.*

Proof of Theorem 1.1. By Proposition 2.2, we may assume that X is a bounded definable C^rG submanifold of a representation Ω of G such that $\overline{X_1}-X_1=\cdots=\overline{X_n}-X_n=\overline{X}-X=\{0\}$.

Let $f: X \to \mathbb{R}, f(x)=||x||^{-1}$, where $||x||$ denotes the standard norm of x in Ω. Since f is submersive and G invariant and by Theorem 2.1, there exist a sufficiently large positive number α and a definable C^rG map $h_1: f^{-1}((\alpha,\infty)) \to f^{-1}(\alpha)$ such that $h := (f,h_1): f^{-1}((\alpha,\infty)) \to (\alpha,\infty)\times f^{-1}(\alpha)$ is a definable C^rG diffeomorphism.

Let $f_i := f|X_i$. Since $(X;X_1,\dots,X_n)$ satisfies the frontier condition and $X_1,\dots,X_n$ are in general position, each $f_i^{-1}((\alpha,\infty))$ is a definable C^rG submanifold of $f^{-1}((\alpha,\infty))$, $f_1^{-1}((\alpha,\infty)),\dots,f_n^{-1}((\alpha,\infty))$ are in general position in $f^{-1}((\alpha,\infty))$ and every $f_i|f_i^{-1}((\alpha,\infty)): f_i^{-1}((\alpha,\infty)) \to (\alpha,\infty)$ is a surjective submersive definable C^rG map. Moreover $f_1^{-1}(\alpha),\dots,f_n^{-1}(\alpha)$ are in general position in $f^{-1}(\alpha)$.

Assertion. (1) There exists a definable C^rG map $h_2: f^{-1}((\alpha,\infty)) \to f^{-1}(\alpha)$ such that for any i, $h_2(f_i^{-1}((\alpha,\infty))) \subset f_i^{-1}(\alpha)$ and h_2 is an approximation of h_1.

(2) Let Y, Z be affine definable C^rG manifolds, $Y_1,\dots,Y_n$ (resp. $Z_1,\dots,Z_n$) definable C^rG submanifolds of Y (resp. Z) in general position. $F: (\cup_{i=1}^n Y_i; Y_1,\dots,Y_n) \to (\cup_{i=1}^n Z_i; Z_1,\dots,Z_n)$ be a definable G map. If each $F|Y_i$ is a definable C^rG map $(Y_i; Y_i\cap Y_1,\dots,Y_i\cap Y_{i-1}, Y_i\cap Y_{i+1},\dots,Y_i\cap Y_n) \to (Z_i; Z_i\cap Z_1,\dots,Z_i\cap Z_{i-1},Z_i\cap Z_{i+1},\dots,Z_i\cap Z_n)$, then there exist a G invariant definable open neighborhood W_n of $\cup_{i=1}^n Y_i$ in Y and a definable C^rG map $\Psi: (W_n; Y_1,\dots,Y_n) \to (Z; Z_1,\dots,Z_n)$ such that $\Psi|\cup_{i=1}^n Y_i = F$.

Assertion (2) is proved by Assertion in the proof of Theorem 1.3 without using Theorem 1.1.

We now prove Assertion (1) by induction on n. If $n=0$, then Assertion (1) is trivial.

Let $n \geq 1$. Since $f_1^{-1}((\alpha,\infty)),\dots,f_n^{-1}((\alpha,\infty))$ are in general position in $f^{-1}((\alpha,\infty))$, for each i, $f_i^{-1}((\alpha,\infty))\cap f_1^{-1}((\alpha,\infty)),\dots,f_i^{-1}((\alpha,\infty))\cap f_{i-1}^{-1}((\alpha,\infty)), f_i^{-1}((\alpha,\infty))\cap f_{i+1}^{-1}((\alpha,\infty)),\dots,f_i^{-1}((\alpha,\infty))\cap f_n^{-1}((\alpha,\infty))$ are definable C^rG submanifolds of $f_i^{-1}((\alpha,\infty))$.

By the inductive hypothesis of Assertion (1), for any i, there exists a definable C^rG map $\phi_i: f_i^{-1}((\alpha,\infty)) \to f_i^{-1}(\alpha)$ such that for each j with $j\neq i$, $\phi_i(f_i^{-1}((\alpha,\infty))\cap f_j^{-1}((\alpha,\infty))) \subset (f_i|f_j^{-1}((\alpha,\infty)))^{-1}(\alpha)$ and ϕ_i is an approximation of $h_1|f_i^{-1}((\alpha,\infty))$.

Applying Assertion (2) to $\phi_1|f_2^{-1}((\alpha,\infty))$, there exist a G invariant definable open neighborhood W_1 of $f_2^{-1}((\alpha,\infty))\cap f_1^{-1}((\alpha,\infty))$ in $f_2^{-1}((\alpha,\infty)))$ and a definable C^rG map $\psi_1: (W_1; f_2^{-1}((\alpha,\infty))\cap f_1^{-1}((\alpha,\infty)), f_2^{-1}((\alpha,\infty))\cap f_3^{-1}((\alpha,\infty))\cap f_1^{-1}((\alpha,\infty)),\dots,f_2^{-1}((\alpha,\infty))\cap f_n^{-1}((\alpha,\infty))\cap f_1^{-1}((\alpha,\infty))) \to (f_2^{-1}(\alpha); f_2^{-1}(\alpha)\cap f_1^{-1}(\alpha), f_2^{-1}(\alpha)\cap f_3^{-1}(\alpha)\cap f_1^{-1}(\alpha),\dots,f_2^{-1}(\alpha)\cap f_n^{-1}(\alpha)\cap f_1^{-1}(\alpha))$. Take a G invariant definable open neighborhood $W_1' \subset W_1$ of $f_2^{-1}((\alpha,\infty)))\cap f_1^{-1}((\alpha,\infty))$ in $f_2^{-1}((\alpha,\infty)))$ whose closure in $f_2^{-1}((\alpha,\infty))$ is properly contained in W_1 and a G invariant definable C^r function $a: f_2^{-1}((\alpha,\infty)) \to \mathbb{R}$ such that its support lies in W_1 and $a|W_1'=1$. By Theorem 2.4, we have a G invariant definable open neighborhood O of $f_2^{-1}(\alpha)$ in Ω and a definable C^rG map $\theta_{f_2^{-1}(\alpha)}: O \to f_2^{-1}(\alpha)$ with $\theta|f_2^{-1}(\alpha) = id_{f_2^{-1}(\alpha)}$.

Define $\psi_2'(x) =$

$$\begin{cases} \theta_{f_2^{-1}(\alpha)}((1-a(x))\phi_2(x)+a(x)\psi_1(x)), & x\in W_1 \\ \phi_2(x), & x\in f_2^{-1}((\alpha,\infty))-W_1 \end{cases}.$$

Then $\psi_2': f_2^{-1}((\alpha,\infty)) \to f_2^{-1}(\alpha)$ is a definable C^rG map which is an approximation of $h_1|f_2^{-1}((\alpha,\infty)))$.

Thus $\phi_1|f_2^{-1}((\alpha,\infty))$ is extensible to a definable G map $\phi_1': f_1^{-1}((\alpha,\infty))\cup f_2^{-1}((\alpha,\infty)) \to f^{-1}(\alpha)$ such that $\phi_1'|f_1^{-1}((\alpha,\infty)), \phi_1'|f_2^{-1}((\alpha,\infty))$ are definable C^rG maps and $\phi_1'|f_1^{-1}((\alpha,\infty)) \subset f_1^{-1}(\alpha), \phi_1'|f_2^{-1}((\alpha,\infty)) \subset f_2^{-1}(\alpha)$.

Repeating this process, we have a definable G map $\Phi: (\cup_{i=1}^n f_i^{-1}((\alpha,\infty)); f_1^{-1}((\alpha,\infty)),\dots,f_n^{-1}((\alpha,\infty))) \to (f^{-1}(\alpha); f_1^{-1}(\alpha),\dots,f_n^{-1}(\alpha))$ such that each $\Phi|f_i^{-1}((\alpha,\infty))$ is a definable C^rG map which is an approximation of $h_1|f_i^{-1}((\alpha,\infty))$.

By Assertion (2), we have a G invariant definable open neighborhood U of $\cup_{i=1}^n f_i^{-1}(($

$\alpha,\infty)$) in $f^{-1}((\alpha,\infty))$ and a definable C^rG map $F : U \to f^{-1}(\alpha)$ extending Φ.

Take a G invariant definable open neighborhood U' of $\cup_{i=1}^n f_i^{-1}((\alpha,\infty))$ in $f^{-1}((\alpha,\infty))$ whose closure in $f^{-1}((\alpha,\infty))$ is properly contained in U and a G invariant definable C^r function $b : f^{-1}((\alpha,\infty)) \to \mathbb{R}$ such that its support lies in U and $b|U' = 1$.

By Theorem 2.4, we have a G invariant definable open neighborhood V of $f^{-1}(\alpha)$ in Ω and a definable C^rG map $\theta_{f^{-1}(\alpha)} : V \to f^{-1}(\alpha)$ with $\theta_{f^{-1}(\alpha)}|f^{-1}(\alpha) = id_{f^{-1}(\alpha)}$.

Define $h_2(x) =$

$$\begin{cases} \theta_{f^{-1}(\alpha)}((1-b(x))h_1(x) + b(x)F(x)), & x \in U \\ h_1(x), & x \in f^{-1}((\alpha,\infty)) - U \end{cases}.$$

Then $h_2 : f^{-1}((\alpha,\infty)) \to f^{-1}(\alpha)$ is the required definable C^rG map and the proof of Assertion (1) is complete.

Since h_2 is an approximation of h_1 and Theorem 1.4, $H := (f, h_2) : (f^{-1}((\alpha,\infty)); f_1^{-1}((\alpha,\infty)),\dots,f_n^{-1}((\alpha,\infty))) \to (\alpha,\infty) \times (f^{-1}(\alpha); f_1^{-1}(\alpha),\dots,f_n^{-1}(\alpha))$ is a definable C^rG diffeomorphism. If α is sufficiently large, then $f^{-1}([0,\alpha+1])$ is a compact definable C^rG manifold with boundary $f^{-1}(\alpha+1)$ and each $f_i^{-1}([0,\alpha+1])$ is a compact definable C^rG submanifold of $f^{-1}([0,\alpha+1])$ with boundary $f_i^{-1}(\alpha+1)$. Therefore using H, $(X; X_1,\dots,X_n)$ is definably C^rG diffeomorphic to $(f^{-1}([0,\alpha+1]); f_1^{-1}([0,\alpha+1]),\dots,f_n^{-1}([0,\alpha+1]))$. □

Proof of Theorem 1.2. By induction on n, we simultaneously prove the theorem and the following assertion.

Assertion. Let $f : \cup_{i=1}^n \partial X_i \times [0,1] \to \cup_{i=1}^n X_i$ ($\subset X$) be a definable G map. If each $f|\partial X_i \times [0,1]$ is a relative definable C^rG collar of $(\partial X_i; \partial X_i \cap \partial X_1,\dots,\partial X_i \cap \partial X_{i-1}, \partial X_i \cap \partial X_{i+1},\dots,\partial X_i \cap \partial X_n)$ in $(X_i; X_i \cap X_1,\dots,X_i \cap X_{i-1}, X_i \cap X_{i+1},\dots,X_i \cap X_n)$, then there exists a positive number ϵ such that $f| \cup_{i=1}^n \partial X_i \times [0,\epsilon]$ is extensible to a relative definable C^rG collar $\phi : (\partial X; \partial X_1,\dots,\partial X_n) \times [0,\epsilon] \to (X; X_1,\dots,X_n)$ of $(\partial X; \partial X_1,\dots,\partial X_n)$ in $(X; X_1,\dots,X_n)$.

If $n = 0$, then the theorem is proved by Theorem 2.3 and Assertion is trivial.

Let $n \geq 1$. By the inductive hypothesis of Theorem 1.2, we can find a relative definable C^rG collar $(\partial X_1; \partial X_1 \cap \partial X_2,\dots,\partial X_1 \cap \partial X_n) \times [0,1] \to (X_1; X_1 \cap X_2,\dots,X_1 \cap X_n)$ of $(\partial X_1; \partial X_1 \cap \partial X_2,\dots,\partial X_1 \cap \partial X_n)$ in $(X_1; X_1 \cap X_2,\dots,X_1 \cap X_n)$. Applying the inductive hypothesis of Assertion, one has a positive number ϵ' and a definable G map $\tilde{\phi} : \cup_{i=1}^n \partial X_i \times [0,\epsilon'] \to \cup_{i=1}^n X_i$ such that each $\tilde{\phi}|\partial X_i \times [0,\epsilon']$ is a relative definable C^rG collar of $(\partial X_i; \partial X_i \cap \partial X_1,\dots,\partial X_i \cap \partial X_{i-1}, \partial X_i \cap \partial X_{i+1},\dots,\partial X_i \cap \partial X_n)$ in $(X_i; X_i \cap X_1,\dots,X_i \cap X_{i-1}, X_i \cap X_{i+1},\dots,X_i \cap X_n)$. After composing $id \times f_{\epsilon'}$, we may assume that the domain of $\tilde{\phi}$ is $\cup_{i=1}^n \partial X_i \times [0,1]$, where $f_{\epsilon'}$ denotes a definable C^ω diffeomorphism from $[0,1]$ onto $[0,\epsilon']$.

We now extend $\tilde{\phi}$ to a definable C^rG map $\overline{\phi} : U \times [0,1] \to X$, where U is a G invariant definable open neighborhood of $\cup_{i=1}^n \partial X_i$ in ∂X.

Let Ω be a representation of G containing X as a closed definable C^rG manifold. By Theorem 2.4, we can take a definable C^rG tubular neighborhood (U_X, θ) of X in Ω. If $n = 1$, then the composition $\tilde{\phi}$ and θ is the required extension.

Let $n > 1$. By the inductive hypothesis, there exist G invariant definable open neighborhoods $U_{n-1} \subset U'_{n-1}$ of $\cup_{i=1}^{n-1} \partial X_i$ in ∂X, a G invariant definable open neighborhood of U_n of ∂X_n in ∂X, and definable C^rG maps $f_{n-1} : U'_{n-1} \times [0,1] \to X \subset \Omega$, $f_n : U_n \times [0,1] \to X \subset \Omega$ such that the closure of U_{n-1} in ∂X is properly contained in U'_{n-1}, $f_{n-1}|(\cup_{i=1}^{n-1} \partial X_i \times [0,1]) = \tilde{\phi}|(\cup_{i=1}^{n-1} \partial X_i \times [0,1])$ and $f_n|\partial X_n \times [0,1] = \tilde{\phi}|\partial X_n \times [0,1]$. Take a G invariant definable C^r function h on $U_n \times [0,1]$ whose support lies in $(U_n \cap U'_{n-1}) \times [0,1]$ with $h|(U_n \cap U_{n-1}) \times [0,1] = 1$. Then $hf_{n-1}|(U_n \cap U'_{n-1}) \times [0,1]$ is extensible to a definable C^rG map $\overline{f_{n-1}}$ defined on $U_n \times [0,1]$. Let $U := U_{n-1} \cup U_n$. Then $f_{n-1}|U_{n-1} \times [0,1]$ is extensible to a definable C^rG map f'_{n-1} defined on $U \times [0,1]$. Take a G invariant definable C^r function $\overline{h}$ on $U \times [0,1]$ such that $\overline{h} = 1$ on some G invariant definable open neighborhood of $\partial X_n \times [0,1]$ in $U_n \times [0,1]$ and its support lies in $U_n \times [0,1]$. Define $\overline{\phi} : U \times [0,1] \to X, \overline{\phi}(x) =$

$$\begin{cases} \theta((1-\overline{h}(x))f'_{n-1}(x) + \overline{h}(x)f_n(x)), & \\ \qquad\qquad x \in U_n \times [0,1] \\ f'_{n-1}(x), \qquad x \in (U - U_n) \times [0,1] \end{cases}.$$

Then $\overline{\phi}$ is the required extension.

We now construct a relative definable C^r G collar $\phi : (\partial X; \partial X_1, \dots, \partial X_n) \times [0,1] \to (X; X_1, \dots, X_n)$ as an extension of $\tilde{\phi}$. Let $V \subset U$ be a G invariant definable open neighborhood of $\cup_{i=1}^n \partial X_i$ whose closure in ∂X is properly contained in U and let ψ be a G invariant definable C^r function on $\partial X \times [0,1]$ such that its support lies in $U \times [0,1]$ and $\psi | V \times [0,1] = 1$. By Theorem 2.3, we have a definable $C^r G$ collar $\phi' : \partial X \times [0,1] \to X$ of ∂X in X. Then $\phi : (\partial X; \partial X_1, \dots, \partial X_n) \to (X; X_1, \dots, X_n)$ defined by $\phi(x) =$

$$\begin{cases} \theta((1-\psi(x))\phi'(x) + \psi(x)\overline{\phi}(x)), & \\ \qquad\qquad x \in U \times [0,1] \\ \phi'(x), \qquad x \in (\partial X - U) \times [0,1] \end{cases}$$

is a relative definable $C^r G$ collar of $(\partial X; \partial X_1, \dots, \partial X_n)$ in $(X; X_1, \dots, X_n)$ such that $\phi | (\cup_{i=1}^n \partial X_i) \times [0,1] = \tilde{\phi}$. □

The following is an approximation theorem in the equivariant definable category.

Theorem 2.5 (1.1 [9]). *If G is a compact definable C^r group and $1 \le s < r < \infty$, then every definable $C^s G$ map between affine definable $C^r G$ manifolds is approximated by a definable $C^r G$ map in the definable C^s topology.*

Proof of Theorem 1.3. Since X, Y are affine, we may assume that they are definable $C^r G$ submanifolds of some representation Ω of G.

We simultaneously prove the theorem and the following assertion by induction on n.

Assertion. Let $F : (\cup_{i=1}^n X_i; X_1, \dots, X_n) \to (\cup_{i=1}^n Y_i; Y_1, \dots, Y_n)$ be a definable G map. If each $F|X_i$ is a definable $C^r G$ map $(X_i; X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n) \to (Y_i; Y_i \cap Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_n)$, then there exist a G invariant definable open neighborhood W_n of $\cup_{i=1}^n X_i$ in X and a definable $C^r G$ map $\phi : (W_n; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_k)$ such that $\phi | \cup_{i=1}^n X_i = F$.

If $n = 0$, then Theorem 2.5 proves the theorem and Assertion is trivial.

Let $n \ge 1$. Since $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are in general position, for each i, $X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n$ (resp. $Y_i \cap Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_n$) are definable $C^r G$ submanifolds of X_i (resp. Y_i). Thus $f|X_i : (X_i; X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n) \to (Y_i; Y_i \cap Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_n)$ is a definable $C^1 G$ map. By the inductive hypothesis of Theorem 1.3, we can find a definable $C^r G$ map $f_i : (X_i; X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n) \to (Y_i; Y_i \cap Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_n)$ as an approximation of $f|X_i$.

Applying the inductive hypothesis of Assertion to $f_n | X_1 \cap X_n : X_1 \cap X_n, \to Y_n$, we have a G invariant definable open neighborhood V_1 of $X_1 \cap X_n$ in X_1 and a definable $C^r G$ map $k_1 : V_1 \to Y_n$ such that $k_1 | X_1 \cap X_n = f_n | X_1 \cap X_n$.

Take a smaller G invariant definable open neighborhood $V_1' \subset V_1$ of $X_1 \cap X_n$ in X_1 and a G invariant definable C^r function a_1 on X_1 such that the closure of V_1' in X_1 is properly contained in V_1, the support of a_1 lies in V_1 and $a_1 | V_1' = 1$. By Theorem 2.4, we have a G invariant definable open neighborhood W_1 of Y_1 in Ω and a definable $C^r G$ map $\theta_{Y_1} : W_1 \to Y_1$ with $\theta_{Y_1} | Y_1 = id_{Y_1}$.

Define $k_1' : X_1 \to Y_1, k_1'(x) =$

$$\begin{cases} \theta_{Y_1}((1-a_1(x))f_1(x) + a_1(x)k_1(x)), & \\ \qquad\qquad x \in V_1 \\ f_1(x), \qquad x \in X_1 - V_1 \end{cases}.$$

Then k_1' is a definable $C^r G$ map extending $f_n | X_1 \cap X_n$.

Repeating this process, we have a definable G map $\phi_n : (\cup_{i=1}^n X_i; X_1, \dots, X_n) \to (\cup_{i=1}^n Y_i; Y_1, \dots, Y_n)$ such that each $\phi_n | X_i$ is a definable $C^r G$ map which is an approximation of $f|X_i$.

By the inductive hypothesis of Assertion, there exist G invariant definable open neighborhood U_{n-1} of $\cup_{i=1}^{n-1} X_i$ in X and a definable C^rG map $f'_{n-1} : (U_{n-1}; X_1, \dots, X_{n-1}) \to (Y; Y_1, \dots, Y_{n-1})$ such that $f'_{n-1} | \cup_{i=1}^{n-1} X_i = \phi_n | \cup_{i=1}^{n-1} X_i$.

By Theorem 2.4, $\phi_n | X_n$ is extensible to a definable C^rG map F_n from a G invariant definable open neighborhood U_n of X_n in X, and we have a G invariant definable open neighborhood V of Y in Ω and a definable C^rG map $\theta_Y : V \to Y$ with $\theta_Y | Y = id_Y$.

Take a smaller G invariant definable open neighborhood $U'_n \subset U_n$ of X_n of X and a G invariant definable C^r function $b : X \to \mathbb{R}$ such that the closure of U'_n in X is properly contained in U_n, its support lies in U_n and $b | U'_n = 1$.

Define $H_n : U_{n-1} \cup U_n \to Y, H_n(x) =$

$$\begin{cases} \theta_Y((1-b(x))f'_{n-1}(x) + b(x)F_n(x)), \\ \qquad\qquad\qquad\qquad\qquad x \in U_n \\ f'_{n-1}(x), \quad x \in U_{n-1} - U_n \end{cases}.$$

Then H_n is a definable C^rG map. Since $F_n | (X_n \cap (\cup_{i=1}^{n-1} X_i)) = \phi_n | (X_n \cap (\cup_{i=1}^{n-1} X_i)) = f'_{n-1} | (X_n \cap (\cup_{i=1}^{n-1} X_i))$, H_n is a definable C^rG map $(U_{n-1} \cup U_n; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ and Assertion is proved.

Take a G invariant definable open neighborhood $\tilde{U}_n$ of $\cup_{i=1}^n X_i$ in X whose closure in X is properly contained in $U_{n-1} \cup U_n$ and a G invariant definable C^r function $c : X \to \mathbb{R}$ such that its support lies in $U_{n-1} \cup U_n$ and $c | \tilde{U} = 1$.

Applying Theorem 2.5 to $f : X \to Y$, there exists a definable C^rG map $\tilde{f} : X \to Y$ as an approximation of $f : X \to Y$.

Define $h(x) =$

$$\begin{cases} \theta_Y((1-c(x))\tilde{f}(x) + c(x)H_n(x)), \\ \qquad\qquad\qquad\qquad x \in U_{n-1} \cup U_n \\ \tilde{f}(x), \quad x \in X - U_{n-1} \cup U_n \end{cases}.$$

Then h is the required definable C^rG map. □

By a way similar to the proof of Theorem 1.3 proves the following stronger version.

Theorem 2.6. *Let X, Y be affine definable C^rG manifolds and $X_1, \dots, X_n$ (reps. $Y_1, \dots, Y_n$) definable C^rG submanifolds of X (resp. Y) such that $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$) are in general position. Suppose that $f : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ is a definable C^sG map and $1 \le s < r < \infty$. Then f is approximated by a definable C^rG map $h : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ in the definable C^s topology. Moreover if for $1 \le i_1 < \dots < i_k \le n$, $f|X_{i_1}, \dots, f|X_{i_k}$ are definable C^rG maps, then we can take h such that $h | \cup_{j=1}^k X_{i_j} = f | \cup_{j=1}^k X_{i_j}$.*

A subset V of $\mathbb{R}^n$ is an *algebraic subset* of $\mathbb{R}^n$ if it is the zero set of some polynomial function on $\mathbb{R}^n$. An *algebraic set* means an algebraic subset of some $\mathbb{R}^n$. A point x in an algebraic set $V \subset \mathbb{R}^n$ is called *nonsingular of dimension d in V* if there exist polynomial functions $p_i : \mathbb{R}^n \to \mathbb{R}$, $(1 \le i \le n-d)$, and an open neighborhood U of x in $\mathbb{R}^n$ such that:

1. $p_i(V) = 0$, $(1 \le i \le n-d)$.
2. $V \cap U = U \cap (\cap_{i=1}^{n-d} p_i(0))$.
3. The gradients $(\nabla p_i)_x$ $(1 \le i \le n-d)$ are linearly independent on U.

The *dimension* $\dim V$ of V is $\max\{d |$there exists an $x \in V$ which is nonsingular of dimension $d\}$. *Nonsing* $V = \{x \in V | x$ is nonsingular of dimension $\dim V\}$ and *Sing* $V = V -$ *Nonsing* V. An algebraic set is *nonsingular* if *Sing* $V = \emptyset$. Remark that *Sing* V is an algebraic subset of V with dim *Sing* $V <$ dim V. An algebraic subset W of a nonsingular algebraic set V is a *nonsingular algebraic subset* of V if W is nonsingular.

Theorem 2.7 ([1])**.** *Let X be a compact C^∞ manifold and $X_1, \dots, X_n$ compact C^∞ submanifolds of X in general position. Then there exist a nonsingular algebraic set Y and a C^∞ diffeomorphism $\phi : X \to Y$ such that each $\phi(X_i)$ is a nonsingular algebraic subset Y_i of Y. In particular, $(X; X_1, \dots, X_n)$ admits a simultaneous Nash manifold structure $(Y; Y_1, \dots, Y_n)$.*

Some refinement of the proof of 2.2.9 [7] proves the following relative version of it.

Theorem 2.8. *Let X be a compact C^s manifold and $X_1, \dots, X_n$ compact C^s submanifolds of X in general position and $1 \leq s < \infty$. Then there exist a compact C^∞ manifold Y and its compact C^∞ submanifolds $Y_1, \dots, Y_n$ of Y such that $(X; X_1, \dots, X_n)$ is C^s diffeomorphic to $(Y; Y_1, \dots, Y_n)$.*

Theorem 2.9 (1.1 [11]). *Every definable C^r manifold is affine.*

By Theorem 2.9, the polynomial approximation theorem, the set of C^r diffeomorphisms is dense in the set of C^1 diffeomorphisms in the C^1 Whitney topology and by a way similar to the proof of Theorem 1.3, we have the following result.

Theorem 2.10. *Let X, Y be compact definable C^r manifolds and $X_1, \dots, X_n$ (reps. $Y_1, \dots, Y_n$) compact definable C^r submanifolds of X (resp Y) such that $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$) are in general position. Suppose that $f : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ is a C^1 diffeomorphism. Then there exists a definable C^r diffeomorphism $h : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ as an approximation of f in the C^1 Whitney topology.*

Proof of Theorem 1.5. By Theorem 2.9, X is affine.

Assume that $X, X_1, \dots, X_n$ are compact. By Theorem 2.8, there exist compact C^∞ manifold X' and compact C^∞ submanifolds $X'_1, \dots, X'_n$ of X' such that $(X; X_1, \dots, X_n)$ is C^2 diffeomorphic to $(X'; X'_1, \dots, X'_n)$. Thus by Theorem 2.7, we can find a nonsingular algebraic set $\tilde{X}$ and nonsingular algebraic subsets $\tilde{X}_1, \dots, \tilde{X}_n$ of $\tilde{X}$ such that $(X'; X'_1, \dots, X'_n)$ is C^∞ diffeomorphic to $(\tilde{X}; \tilde{X}_1, \dots, \tilde{X}_n)$. Hence $(X; X_1, \dots, X_n)$ is C^2 diffeomorphic to $(\tilde{X}; \tilde{X}_1, \dots, \tilde{X}_n)$. By Theorem 2.10 and since $X, X_1, \dots, X_n$ are compact, $(X; X_1, \dots, X_n)$ is definably C^2 diffeomorphic to $(\tilde{X}; \tilde{X}_1, \dots, \tilde{X}_n)$.

Assume that $X, X_1, \dots, X_n$ are noncompact and $(X; X_1, \dots, X_n)$ satisfies the frontier condition. By Theorem 1.1, there exist a compact definable C^2 manifold Y with boundary ∂Y, compact definable C^2 submanifolds $Y_1, \dots, Y_n$ of Y with boundary $\partial Y_1, \dots, \partial Y_n$, respectively, and a definable C^2 diffeomorphism $f : X \to Int\, Y$ such that $f(X_i) = Int\, Y_i$, each ∂Y_i is contained in ∂Y, and $Y_1, \dots, Y_n$ and ∂Y are in general position in Y. Thus by Theorem 1.2, $(Y; Y_1, \dots, Y_n)$ admits a relative definable C^2 collar. Hence we have the relative definable C^2 double $(D; D_1, \dots, D_n)$ of $(Y; Y_1, \dots, Y_n)$. Note that $D, D_1, \dots, D_n$ and ∂Y are compact and $D_1, \dots, D_n, \partial Y$ are in general position.

By the argument in the first case, there exist a definable C^r manifold W and definable C^r submanifolds $W_1, \dots, W_n, U$ of W such that $(D; D_1, \dots, D_n, \partial Y)$ is definably C^2 diffeomorphic to $(W; W_1, \dots, W_n, U)$. Therefore we can find some unions $\tilde{X}, \tilde{X}_1, \dots, \tilde{X}_n$ of connected components of $W - U, W_1 - U, \dots, W_n - U$, respectively, such that $\tilde{X}$ is a definable C^r manifold, each $\tilde{X}_i$ is a definable C^r submanifold of $\tilde{X}$ and $(X; X_1, \dots, X_n)$ is definably C^2 diffeomorphic to $(\tilde{X}, \tilde{X}_1, \dots, \tilde{X}_n)$.

The latter half follows from Theorem 1.3 and Theorem 1.4. □

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