Relative properties of definable C^∞ manifolds with finite abelian group actions in an o-minimal expansion of R_{exp}

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Abstract

Let G be a finite abelian group and $\mathcal{M}$ an o-minimal expansion of $\mathbf{R}_{exp} = (\mathbb{R}, +, \cdot, <, e^x)$ admitting the C^∞ cell decomposition. Everything is considered in $\mathcal{M}$.

We prove that every definable $C^\infty G$ manifold is affine. Moreover we prove that if $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$) are definable $C^\infty G$ submanifolds of a definable $C^\infty G$ manifold X (resp. Y) in general position, then every definable $C^1 G$ map $(X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ is approximated by a definable $C^\infty G$ map $(X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$.

Furthermore we prove a relative collaring theorem.

2000 *Mathematics Subject Classification.* 14P10, 14P20, 57S17, 58A05, 03C64.
Keywords and Phrases. O-minimal, definable $C^\infty G$ manifolds, affineness, simultaneous definable $C^\infty G$ approximations, relative collaring theorem.

1 . Introduction.

Let $\mathbf{R}_{exp}$ denote $(\mathbb{R}, +, \cdot, <, e^x)$ is the exponential field expanding the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. Then $\mathbf{R}_{exp}$ is o-minimal [2]. Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, e^x, \dots)$ be an o-minimal expansion of $\mathbf{R}_{exp}$ admitting the C^∞ cell decomposition.

General references on o-minimal structures are [1], [3], [16], and it is known in [14] that there exist uncountably many o-minimal expansions of $\mathcal{R}$. For example, the Nash category is a special case of the definable C^∞ category and it coincides with the definable C^∞ category based on $\mathcal{R}$ [17]. Further properties and constructions of them are studied in [2], [4], [13]. Equivariant definable category is studied in [6], [7], [8], [10].

In this paper "definable" means "definable with parameters in $\mathcal{M}$", everything is considered in $\mathcal{M}$, every manifold does not have boundary, and each definable map is continuous unless otherwise stated.

Theorem 1.1. *If G is a finite abelian group, then every definable $C^\infty G$ manifold is affine.*

In Theorem 1.1, we cannot drop the assumption that $\mathcal{M}$ is an o-minimal expansion of $\mathbf{R}_{exp}$. Even if G is trivial, there exist uncountably many nonaffine Nash manifolds [15]. If $r < \infty$, then the non-equivariant definable C^r case is proved without this assumption [9].

Let X be a C^∞ manifold and $X_1, \dots, X_n$ C^∞ submanifolds of X. We say that $X_1, \dots, X_n$ *are in general position* in X if for each $i \in \{1, \dots, n\}$ and $J \subset \{1, \dots, n\} - \{i\}$, X_i intersects transverse to $\cap_{j \in J} X_j$.

Let G be a compact definable C^∞ group. Let X be a definable $C^\infty G$ manifold with boundary ∂X and $X_1, \dots, X_n$ definable $C^\infty G$ submanifolds of X with boundary $\partial X_1, \dots, \partial X_n$, respectively, such that every ∂X_i is contained in ∂X. A *relative definable $C^\infty G$ collar* of $(\partial X; \partial X_1, \dots, \partial X_n)$ is a definable $C^\infty G$ imbedding $\phi : (\partial X \times [0,1]; \partial X_1 \times [0,1], \dots, \partial X_n \times [0,1]) \to (X; X_1, \dots, X_n)$ such that $\phi | \partial X \times \{0\}$ is the inclusion $\partial X \to X$, where the action on $[0,1]$ is trivial.

Theorem 1.2. *Let G be a finite abelian group. Let X be a compact definable $C^\infty G$ manifold with boundary ∂X, and $X_1, \dots, X_n$ compact definable $C^\infty G$ submanifolds of X with boundary $\partial X_1, \dots, \partial X_n$, respectively, such that $X_1, \dots, X_n, \partial X$ are in general position and every ∂X_i is contained in ∂X. Then there exists a relative definable $C^\infty G$ collar $\phi : (\partial X \times [0,1]; \partial X_1 \times [0,1], \dots, \partial X_n \times [0,1]) \to (X; X_1, \dots, X_n)$ of $(\partial X; \partial X_1, \dots, \partial X_n)$.*

If $r < \infty$, G is a compact definable C^r group and X is affine, then the definable $C^r G$ case of Theorem 1.2 is proved without the assumption that $\mathcal{M}$ is an o-minimal expansion of $\mathbf{R}_{exp}$ [12]. Theorem 1.2 is a relative definable C^∞ version of 4.6 [7].

Let $Def^r(\mathbb{R}^n)$ denote the set of definable C^r functions on $\mathbb{R}^n$. For every $f \in Def^r(\mathbb{R}^n)$ and for every positive definable function $\epsilon : \mathbb{R}^n \to \mathbb{R}$, the ϵ-neighborhood $N(f;\epsilon)$ of f in $Def^r(\mathbb{R}^n)$ is defined by $\{h \in Def^r(\mathbb{R}^n) | |\partial^\alpha (h-f)| < \epsilon, \forall \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \le r\}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n, |\alpha| = \alpha_1 + \dots + \alpha_n, \partial^\alpha F = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. We call the topology defined by these ϵ-neighborhoods the *definable C^r topology*. By considering relative topology, we can define the *definable C^r topology* on a definable C^r submanifold of $\mathbb{R}^n$.

The following is a relative definable $C^\infty G$ version of 1.1 [7].

Theorem 1.3. *Let G be a finite abelian group. Let X, Y be definable $C^\infty G$ manifolds and $X_1, \dots, X_n$ (reps. $Y_1, \dots, Y_n$) definable $C^\infty G$ submanifolds of X (resp Y) such that $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$) are in general position. Suppose that $f : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ is a definable $C^1 G$ map. Then f is approximated by a definable $C^\infty G$ map $h : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ in the definable C^1 topology. Moreover if for $1 \le i_1 < \dots < i_k \le n$, $f|X_{i_1}, \dots, f|X_{i_k}$ are definable $C^\infty G$ maps, then we can take h such that $h| \cup_{j=1}^k X_{i_j} = f | \cup_{j=1}^k X_{i_j}$.*

If $r < \infty$, G is a compact definable C^r group and X is affine, then without the assumption that $\mathcal{M}$ is an o-minimal expansion of $\mathbf{R}_{exp}$, the definable $C^r G$ case of Theorem 1.3 is proved [12].

2 Proof of our results

The following result is affineness of definable C^∞ manifolds.

Theorem 2.1 ([5])**.** *Every definable C^∞ manifold of dimension n is definably C^∞ imbeddable into $\mathbb{R}^{2n+1}$.*

Let G be a compact definable C^∞ group. A *representation map* of G is a group homomorphism from G to some orthogonal group which is a definable C^∞ map. A *representation* means the representation space of a representation map of G. In this paper, we assume that every representation of G is orthogonal.

Definable $C^\infty G$ manifolds are studied in [8], [10], [11]. A *definable $C^\infty G$ submanifold* of a representation Ω of G is a G invariant definable C^∞ submanifold of Ω. A definable $C^\infty G$ manifold is *affine* if it is definably $C^\infty G$ diffeomorphic to a definable $C^\infty G$ submanifold of some representation of G. By [10], if G is a compact affine definable C^∞ group, then every compact definable $C^\infty G$ manifold is affine.

Proof of Theorem 1.1. Let $G = \{g_1, \dots, g_m\}$ and X a definable $C^\infty G$ manifold of dimension n. By Theorem 2.1, there exists a definable C^∞ imbedding $f : X \to \mathbb{R}^{2n+1}$. Let Ω be the representation of G whose underlying space is $\mathbb{R}^{(2n+1)m} = \mathbb{R}^{2n+1} \times \cdots \times \mathbb{R}^{2n+1}$ and its action is defined by the permutation of coordinates $(x_1, \dots, x_m) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(m)})$ induce from $(gg_1, \dots gg_m) = (g_{\sigma(1)}, \dots, g_{\sigma(m)})$. Then $F : X \to \Omega, F(x) = (f(g_1 x), \dots, f(g_m x))$ is the required definable $C^\infty G$ imbedding. $\square$

Theorem 2.2 (2.24 [8])**.** *Let G be a compact definable C^∞ group, X a compact affine definable $C^\infty G$ manifold with boundary ∂X. Then there exists a definable $C^\infty G$ collar, namely there exists a definable $C^\infty G$ imbedding $\phi : \partial X \times [0,1] \to X$ such that $\phi|(\partial X \times \{0\})$ is the inclusion $\partial X \to X$, where the action on $[0,1]$ is trivial.*

Theorem 2.3 (2.24 [8])**.** *If G is a compact definable C^∞ group, then every definable $C^\infty G$ submanifold X of a representation Ω of G has a definable $C^\infty G$ tubular neighborhood (U, θ) of X in Ω, namely U is a G invariant definable open neighborhood of X in Ω and $\theta : U \to X$ is a definable $C^\infty G$ map with $\theta|X = id_X$.*

Theorem 2.4 ([5])**.** *Let X be a definable C^∞ manifold and A, B definable disjoint closed subsets of X. Then there exists a definable C^∞ function $\phi : X \to \mathbb{R}$ such that $\phi|A = 1$ and $\phi|B = 0$.*

An equivariant version of Theorem 2.4 is the following.

Theorem 2.5. *Let G be a compact definable C^∞ group and X a compact affine definable $C^\infty G$ manifold. Suppose that A, B are G invariant definable disjoint closed subsets of X. Then there exists a G invariant definable C^∞ function $f : X \to \mathbb{R}$ such that $f|A = 1$ and $f|B = 0$.*

Proof. By the assumption, we may assume that X is a definable $C^\infty G$ submanifold of a representation Ω of G. Since G is a compact Lie group, the orbit map $\pi : \Omega \to \Omega/G \subset \mathbb{R}^s$ is a G invariant proper polynomial map. Since X is compact and A, B are closed in X, $\pi(A), \pi(B)$ are closed in $\mathbb{R}^s$. By Theorem 2.4, there exists a definable C^∞ function $\phi : \mathbb{R}^s \to \mathbb{R}$ such that $\phi|\pi(A) = 1$ and $\phi|\pi(B) = 0$. Therefore $f := \phi \circ (\pi|X) : X \to \mathbb{R}$ is the required G definable C^∞ function. $\square$

Proof of Theorem 1.2. By Theorem 1.1, we may assume that X is affine.

We simultaneously prove the theorem and the following assertion by induction on n.

Assertion. Let $f : \cup_{i=1}^n \partial X_i \times [0,1] \to \cup_{i=1}^n X_i$ $(\subset X)$ be a definable G map. If each $f|\partial X_i \times [0,1]$ is a relative definable $C^\infty G$ collar of $(\partial X_i; \partial X_i \cap \partial X_1, \dots, \partial X_i \cap \partial X_{i-1}, \partial X_i \cap \partial X_{i+1}, \dots, \partial X_i \cap \partial X_n)$ in $(X_i; X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n)$, then there exists a positive number ϵ such that $f| \cup_{i=1}^n \partial X_i \times [0, \epsilon]$ is extensible to a relative definable $C^\infty G$ collar $\phi : (\partial X; \partial X_1, \dots, \partial X_n) \times [0, \epsilon] \to (X; X_1, \dots, X_n)$ of $(\partial X; \partial X_1, \dots, \partial X_n)$ in $(X; X_1, \dots, X_n)$.

If $n = 0$, then the theorem is proved by Theorem 2.2 and Assertion is trivial.

Let $n \geq 1$. By the inductive hypothesis of Theorem 1.2, we can find a relative definable $C^\infty G$ collar $(\partial X_1; \partial X_1 \cap \partial X_2, \dots, \partial X_1 \cap \partial X_n) \times [0,1] \to (X_1; X_1 \cap X_2, \dots, X_1 \cap X_n)$ of $(\partial X_1; \partial X_1 \cap \partial X_2, \dots, \partial X_1 \cap \partial X_n)$ in $(X_1; X_1 \cap X_2, \dots, X_1 \cap X_n)$. Applying the inductive hypothesis of Assertion, one has a positive number ϵ' and a definable G map $\tilde{\phi} : \cup_{i=1}^n \partial X_i \times [0, \epsilon'] \to \cup_{i=1}^n X_i$ such that each $\tilde{\phi}|\partial X_i \times [0, \epsilon']$ is a relative definable $C^\infty G$ collar of $(\partial X_i; \partial X_i \cap \partial X_1, \dots, \partial X_i \cap \partial X_{i-1}, \partial X_i \cap \partial X_{i+1}, \dots, \partial X_i \cap \partial X_n)$ in $(X_i; X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n)$. After composing $id \times f_{\epsilon'}$, we may assume that the domain of $\tilde{\phi}$ is $\cup_{i=1}^n \partial X_i \times [0,1]$, where $f_{\epsilon'}$ denotes a definable C^ω diffeomorphism from $[0,1]$ onto $[0, \epsilon']$.

We now extend $\tilde{\phi}$ to a definable $C^\infty G$ map $\overline{\phi} : U \times [0,1] \to X$, where U is a G invariant definable open neighborhood of $\cup_{i=1}^n \partial X_i$ in ∂X.

Let Ω be a representation of G containing X as a definable $C^\infty G$ submanifold. By Theorem 2.3, we can take a definable $C^\infty G$ tubular neighborhood (U_X, θ_X) of X in Ω. If $n = 1$, then $(\theta_X \circ \tilde{\phi}) \times id$ is the required extension.

Let $n > 1$. By the inductive hypothesis, there exist G invariant definable open neighborhoods $U_{n-1} \subset U'_{n-1}$ of $\cup_{i=1}^{n-1} \partial X_i$ in ∂X, a G invariant definable open neighborhood of U_n of ∂X_n in ∂X, and definable $C^\infty G$ maps $f_{n-1} : U'_{n-1} \times [0,1] \to X \subset \Omega$, $f_n : U_n \times [0,1] \to X \subset \Omega$ such that the closure of U_{n-1} in ∂X is properly contained in U'_{n-1}, $f_{n-1}|(\cup_{i=1}^{n-1} \partial X_i \times [0,1]) = \tilde{\phi}|(\cup_{i=1}^{n-1} \partial X_i \times [0,1])$ and $f_n|\partial X_n \times [0,1] = \tilde{\phi}|\partial X_n \times [0,1]$. Take a G invariant definable C^∞ function h on $U_n \times [0,1]$ whose support lies in $(U_n \cap U'_{n-1}) \times [0,1]$ with $h|(U_n \cap U_{n-1}) \times [0,1] = 1$. Then $hf_{n-1}|(U_n \cap U'_{n-1}) \times [0,1]$ is extensible to a definable $C^\infty G$ map $\overline{f_{n-1}}$ defined on $U_n \times [0,1]$. Let $U := U_{n-1} \cup U_n$. Then $f_{n-1}|U_{n-1} \times [0,1]$ is extensible to a definable $C^\infty G$ map f'_{n-1} defined on $U \times [0,1]$. Take a G invariant definable C^∞ function $\overline{h}$ on $U \times [0,1]$ such that $\overline{h} = 1$ on some G invariant definable open neighborhood of $\partial X_n \times [0,1]$ in $U_n \times [0,1]$ and its support lies in $U_n \times [0,1]$. Define $\overline{\phi} : U \times [0,1] \to X, \overline{\phi}(x) =$

$$\begin{cases} \theta_X((1-\overline{h}(x))f'_{n-1}(x) + \overline{h}(x)f_n(x)), \\ \qquad\qquad\qquad\qquad x \in U_n \times [0,1] \\ f'_{n-1}(x), \qquad x \in (U - U_n) \times [0,1] \end{cases}.$$

Then $\overline{\phi}$ is the required extension.

We now construct a relative definable $C^\infty G$ collar $\phi : (\partial X; \partial X_1, \dots, \partial X_n) \times [0,1] \to (X; X_1, \dots, X_n)$ as an extension of $\tilde{\phi}$. Let $V \subset U$ be a G invariant definable open neighborhood of $\cup_{i=1}^n \partial X_i$ whose closure in ∂X is properly contained in U and let ψ be a G invariant definable C^∞ function on $\partial X \times [0,1]$ such that its support lies in $U \times [0,1]$ and $\psi|V \times [0,1] = 1$. By Theorem 2.2, we have a definable $C^\infty G$ collar $\phi' : \partial X \times [0,1] \to X$ of ∂X in X. Then $\phi : (\partial X; \partial X_1, \dots, \partial X_n) \to (X; X_1, \dots, X_n)$ defined by $\phi(x) =$

$$\begin{cases} \theta_X((1-\psi(x))\phi'(x) + \psi(x)\overline{\phi}(x)), \\ \qquad\qquad\qquad\qquad x \in U \times [0,1] \\ \phi'(x), \qquad x \in (\partial X - U) \times [0,1] \end{cases}$$

is a relative definable $C^\infty G$ collar of $(\partial X; \partial X_1, \dots, \partial X_n)$ in $(X; X_1, \dots, X_n)$ such that $\phi|(\cup_{i=1}^n \partial X_i) \times [0,1] = \tilde{\phi}$. □

The proof of Theorem 1.2 also proves the following.

Theorem 2.6. *Let G be a compact definable C^∞ group. Let X be a compact affine definable $C^\infty G$ manifold with boundary ∂X, and $X_1, \dots, X_n$ compact definable $C^\infty G$ submanifolds of X with boundary $\partial X_1, \dots, \partial X_n$, respectively, such that $X_1, \dots, X_n, \partial X$ are in general position and every ∂X_i is contained in ∂X. Then there exists a relative definable $C^\infty G$ collar $\phi : (\partial X \times [0,1]; \partial X_1 \times [0,1], \dots, \partial X_n \times [0,1]) \to (X; X_1, \dots, X_n)$ of $(\partial X; \partial X_1, \dots, \partial X_n)$.*

Theorem 2.7 ([5]). *Let $0 \le r < \infty$. Then every definable C^r map between definable C^∞ manifolds is approximated by a definable C^∞ map in the definable C^r topology.*

To consider an equivariant version of Theorem 2.7, recall the averaging function.

Let $G = \{g_1, \dots, g_m\}$, X an affine definable $C^\infty G$ manifold and Ω a representation of G. Then we define the averaging function $A : C^\infty(X, \Omega) \to C^\infty(X, \Omega)$ by $A(f)(x) = \frac{1}{m} \sum_{i=1}^m g_i^{-1} f(g_i x)$.

Then we have the following proposition.

Proposition 2.8. *(1) If f is a definable C^∞ map, then $A(f)$ is a definable $C^\infty G$ map.*

(2) Let $Def^\infty(X, \Omega)$ (resp. $Def^\infty_G(X, \Omega)$) denote the set of definable C^∞ maps (resp. definable $C^\infty G$ maps) from X to Ω. Then $A|Def^\infty_G(X, \Omega) = id_{Def^\infty_G(X,\Omega)}$ and $A(Def^\infty(X, \Omega)) = Def^\infty_G(X, \Omega)$.

(3) For any non-negative integer r, $A : Def^\infty(X, \Omega) \to Def^\infty(X, \Omega)$ is continuous in the definable C^r topology.

By Theorem 1.1, Theorem 2.3, Theorem 2.7 and Proposition 2.8, we have the following equivariant version.

Theorem 2.9. *Let G be a finite abelian group and $0 \le r < \infty$. Then every definable C^rG map between definable $C^\infty G$ manifolds is approximated by a definable $C^\infty G$ map in the definable C^r topology.*

Proposition 2.10. *Let G be a finite abelian group and X a definable $C^\infty G$ manifold. Suppose that A, B are G invariant definable disjoint closed subsets of X. Then there exists a G invariant definable C^∞ function $f : X \to \mathbb{R}$ such that $f|A = 1$ and $f|B = 0$.*

Proof. By Theorem 2.4, we have a definable C^∞ function $\phi : X \to \mathbb{R}$ such that $\phi|A = 1$ and $\phi|B = 0$. Let $G = \{g_1, \dots, g_m\}$. Then the function $f : X \to \mathbb{R}$ defined by $f(x) = \frac{1}{m}\sum_{i=1}^m \phi(g_i x)$ is the required function. □

Proof of Theorem 1.3. By Theorem 1.1, X, Y are affine. Hence we may assume that they are definable $C^\infty G$ submanifolds of some representation Ω of G.

We simultaneously prove the theorem and the following assertion by induction on n.

Assertion. Let $F : (\cup_{i=1}^n X_i; X_1, \dots, X_n) \to (\cup_{i=1}^n Y_i; Y_1, \dots, Y_n)$ be a definable G map. If each $F|X_i$ is a definable $C^\infty G$ map $(X_i; X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n) \to (Y_i; Y_i \cap Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_n)$, then there exist a G invariant definable open neighborhood W_n of $\cup_{i=1}^n X_i$ in X and a definable $C^\infty G$ map $\phi : (W_n; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_k)$ such that $\phi| \cup_{i=1}^n X_i = F$.

If $n = 0$, then Theorem 2.9 proves the theorem and Assertion is trivial.

Let $n \ge 1$. Since $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are in general position, for each i, $X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n$ (resp. $Y_i \cap Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_n$) are definable $C^\infty G$ submanifolds of X_i (resp. Y_i). Thus $f|X_i : (X_i; X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n) \to (Y_i; Y_i \cap Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_n)$ is a definable C^1G map. By the inductive hypothesis of Theorem 1.3, we can find a definable $C^\infty G$ map $f_i : (X_i; X_i \cap X_1, \dots, X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_n) \to (Y_i; Y_i \cap Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_n)$ as an approximation of $f|X_i$.

Applying the inductive hypothesis of Assertion to $f_n|X_1 \cap X_n : X_1 \cap X_n, \to Y_n$, we have a G invariant definable open neighborhood V_1 of $X_1 \cap X_n$ in X_1 and a definable $C^\infty G$ map $k_1 : V_1 \to Y_n$ such that $k_1|X_1 \cap X_n = f_n|X_1 \cap X_n$.

Take a smaller G invariant definable open neighborhood $V_1' \subset V_1$ of $X_1 \cap X_n$ in X_1 and a G invariant definable C^∞ function a_1 on X_1 such that the closure of V_1' in X_1 is properly contained in V_1, the support of a_1 lies in V_1 and $a_1|V_1' = 1$. By Theorem 2.3, we have a G invariant definable open neighborhood W_1 of Y_1 in Ω and a definable $C^\infty G$ map $\theta_{Y_1} : W_1 \to Y_1$ with $\theta_{Y_1}|Y_1 = id_{Y_1}$.

Define $k_1' : X_1 \to Y_1, k_1'(x) =$

$$\begin{cases} \theta_{Y_1}((1 - a_1(x))f_1(x) + a_1(x)k_1(x)), & x \in V_1 \\ f_1(x), & x \in X_1 - V_1 \end{cases}.$$

Then k_1' is a definable $C^\infty G$ map extending $f_n|X_1 \cap X_n$.

Repeating this process, we have a definable G map $\phi_n : (\cup_{i=1}^n X_i; X_1, \dots, X_n) \to (\cup_{i=1}^n Y_i; Y_1, \dots, Y_n)$ such that each $\phi_n|X_i$ is a definable $C^\infty G$ map which is an approximation of $f|X_i$.

By the inductive hypothesis of Assertion, there exist G invariant definable open neighborhood U_{n-1} of $\cup_{i=1}^{n-1} X_i$ in X and a definable $C^\infty G$ map $f_{n-1}' : (U_{n-1}; X_1, \dots, X_{n-1}) \to (Y; Y_1, \dots, Y_{n-1})$ such that $f_{n-1}'| \cup_{i=1}^{n-1} X_i = \phi_n| \cup_{i=1}^{n-1} X_i$.

By Theorem 2.3, $\phi_n|X_n$ is extensible to a definable C^rG map F_n from a G invariant definable open neighborhood U_n of X_n in X, and we have a G invariant definable open neighborhood V of Y in Ω and a definable $C^\infty G$ map $\theta_Y : V \to Y$ with $\theta_Y|Y = id_Y$.

Take a smaller G invariant definable open neighborhood $U_n' \subset U_n$ of X_n of X and a G invariant definable C^∞ function $b : X \to \mathbb{R}$ such that the closure of U_n' in X is properly contained in U_n, its support lies in U_n and $b|U'_n = 1$.

Define $H_n : U_{n-1} \cup U_n \to Y, H_n(x) =$

$$\begin{cases} \theta_Y((1-b(x))f'_{n-1}(x) + b(x)F_n(x)), \\ \hfill x \in U_n \\ f'_{n-1}(x), \quad x \in U_{n-1} - U_n \end{cases}.$$

Then H_n is a definable $C^\infty G$ map. Since $F_n|(X_n \cap (\cup_{i=1}^{n-1} X_i)) = \phi_n|(X_n \cap (\cup_{i=1}^{n-1} X_i)) = f'_{n-1}|(X_n \cap (\cup_{i=1}^{n-1} X_i))$, H_n is a definable $C^\infty G$ map $(U_{n-1} \cup U_n; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ and Assertion is proved.

Take a G invariant definable open neighborhood $\tilde{U_n}$ of $\cup_{i=1}^n X_i$ in X whose closure in X is properly contained in $U_{n-1} \cup U_n$ and a G invariant definable C^∞ function $c : X \to \mathbb{R}$ such that its support lies in $U_{n-1} \cup U_n$ and $c|\tilde{U} = 1$.

Applying Theorem 2.9 to $f : X \to Y$, there exists a definable $C^\infty G$ map $\tilde{f} : X \to Y$ as an approximation of $f : X \to Y$.

Define $h(x) =$

$$\begin{cases} \theta_Y((1-c(x))\tilde{f}(x) + c(x)H_n(x)), \\ \hfill x \in U_{n-1} \cup U_n \\ \tilde{f}(x), \quad x \in X - U_{n-1} \cup U_n \end{cases}.$$

Then h is the required definable $C^\infty G$ map. □

By a way similar to the proof of Theorem 1.3 proves the following stronger version.

Theorem 2.11. *Let G be a finite abelian group, X, Y definable $C^\infty G$ manifolds and $X_1, \dots, X_n$ (reps. $Y_1, \dots, Y_n$) definable $C^\infty G$ submanifolds of X (resp. Y) such that $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$) are in general position. Suppose that $f : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ is a definable $C^s G$ map and $1 \le s < \infty$. Then f is approximated by a definable $C^\infty G$ map $h : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ in the definable C^s topology. Moreover if for $1 \le i_1 < \cdots < i_k \le n$, $f|X_{i_1}, \dots, f|X_{i_k}$ are definable $C^\infty G$ maps, then we can take h such that $h| \cup_{j=1}^k X_{i_j} = f| \cup_{j=1}^k X_{i_j}$.*

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